CHAPTER 10

10.1. In Fig. 10.4, let $B = 0.2\cos 120\pi t$ T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by $I(t)$ is negligible. Find:

a) $V_{ab}(t)$: Since B is constant over the loop area, the flux is $\Phi = \pi(0.15)^2 B = 1.41 \times 10^{-2}\cos 120\pi t$ Wb. Now, $emf = V_{ba}(t) = -d\Phi/dt = (120\pi)(1.41 \times 10^{-2})\sin 120\pi t$. Then $V_{ab}(t) = -V_{ba}(t) = \underline{-5.33\sin 120\pi t}$ V.

b) $I(t) = V_{ba}(t)/R = 5.33\sin(120\pi t)/250 = \underline{21.3\sin(120\pi t)}$ mA

10.2. In Fig. 10.1, replace the voltmeter with a resistance, R.

a) Find the current I that flows as a result of the motion of the sliding bar: The current is found through

$$I = \frac{1}{R}\oint \mathbf{E}\cdot d\mathbf{L} = -\frac{1}{R}\frac{d\Phi_m}{dt}$$

Taking the normal to the path integral as $\mathbf{a}_z$, the path direction will be counter-clockwise when viewed from above (in the $-\mathbf{a}_z$ direction). The minus sign in the equation indicates that the current will therefore flow *clockwise*, since the magnetic flux is increasing with time. The flux of $\mathbf{B}$ is $\Phi_m = Bdvt$, and so

$$|I| = \frac{1}{R}\frac{d\Phi_m}{dt} = \frac{Bdv}{R}\quad\text{(clockwise)}$$

b) The bar current results in a force exerted on the bar as it moves. Determine this force:

$$\mathbf{F} = \int I d\mathbf{L}\times\mathbf{B} = \int_0^d I dx\mathbf{a}_x\times B\mathbf{a}_z = \int_0^d \frac{Bdv}{R}\mathbf{a}_x\times B\mathbf{a}_z = -\frac{B^2 d^2 v}{R}\mathbf{a}_y\text{ N}$$

c) Determine the mechanical power required to maintain a constant velocity $\mathbf{v}$ and show that this power is equal to the power absorbed by R. The mechanical power is

$$P_m = Fv = \frac{(Bdv)^2}{R}\text{ W}$$

The electrical power is

$$P_e = I^2 R = \frac{(Bdv)^2}{R} = P_m$$

10.3. Given $\mathbf{H} = 300\,\mathbf{a}_z \cos(3 \times 10^8 t - y)$ A/m in free space, find the emf developed in the general $\mathbf{a}_\phi$ direction about the closed path having corners at

a) (0,0,0), (1,0,0), (1,1,0), and (0,1,0): The magnetic flux will be:

$$\Phi = \int_0^1 \int_0^1 300\mu_0 \cos(3 \times 10^8 t - y)\,dx\,dy = 300\mu_0 \sin(3 \times 10^8 t - y)|_0^1$$
$$= 300\mu_0 \left[\sin(3 \times 10^8 t - 1) - \sin(3 \times 10^8 t)\right] \text{ Wb}$$

Then
$$\text{emf} = -\frac{d\Phi}{dt} = -300(3 \times 10^8)(4\pi \times 10^{-7})\left[\cos(3 \times 10^8 t - 1) - \cos(3 \times 10^8 t)\right]$$
$$= \underline{-1.13 \times 10^5 \left[\cos(3 \times 10^8 t - 1) - \cos(3 \times 10^8 t)\right] \text{ V}}$$

b) corners at (0,0,0), $(2\pi,0,0)$, $(2\pi,2\pi,0)$, $(0,2\pi,0)$: In this case, the flux is

$$\Phi = 2\pi \times 300\mu_0 \sin(3 \times 10^8 t - y)|_0^{2\pi} = 0$$

The emf is therefore $\underline{0}$.

10.4. Conductor surfaces are located at $\rho = 1$cm and $\rho = 2$cm in free space. The volume $1\,\text{cm} < \rho < 2\,\text{cm}$ contains the fields $H_\phi = (2/\rho)\cos(6 \times 10^8 \pi t - 2\pi z)$ A/m and $E_\rho = (240\pi/\rho)\cos(6 \times 10^8 \pi t - 2\pi z)$ V/m.

a) Show that these two fields satisfy Eq. (6), Sec. 10.1: Have

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z}\mathbf{a}_\phi = \frac{2\pi(240\pi)}{\rho}\sin(6 \times 10^8 \pi t - 2\pi z)\,\mathbf{a}_\phi = \frac{480\pi^2}{\rho}\sin(6 \times 10^8 \pi t - 2\pi z)\mathbf{a}_\phi$$

Then
$$-\frac{\partial \mathbf{B}}{\partial t} = \frac{2\mu_0(6 \times 10^8)\pi}{\rho}\sin(6 \times 10^8 \pi t - 2\pi z)\,\mathbf{a}_\phi$$
$$= \frac{(8\pi \times 10^{-7})(6 \times 10^8)\pi}{\rho}\sin(6 \times 10^8 \pi t - 2\pi z) = \frac{480\pi^2}{\rho}\sin(6 \times 10^8 \pi t - 2\pi z)\,\mathbf{a}_\phi$$

b) Evaluate both integrals in Eq. (4) for the planar surface defined by $\phi = 0$, 1cm $< \rho <$ 2cm, $0 < z < 0.1$m, and its perimeter, and show that the same results are obtained: we take the normal to the surface as positive $\mathbf{a}_\phi$, so the the loop surrounding the surface (by the right hand rule) is in the negative $\mathbf{a}_\rho$ direction at $z = 0$, and is in the positive $\mathbf{a}_\rho$ direction at $z = 0.1$. Taking the left hand side first, we find

$$\oint \mathbf{E} \cdot d\mathbf{L} = \int_{.02}^{.01} \frac{240\pi}{\rho}\cos(6 \times 10^8 \pi t)\,\mathbf{a}_\rho \cdot \mathbf{a}_\rho\,d\rho$$
$$+ \int_{.01}^{.02} \frac{240\pi}{\rho}\cos(6 \times 10^8 \pi t - 2\pi(0.1))\,\mathbf{a}_\rho \cdot \mathbf{a}_\rho\,d\rho$$
$$= 240\pi \cos(6 \times 10^8 \pi t)\ln\left(\frac{1}{2}\right) + 240\pi \cos(6 \times 10^8 \pi t - 0.2\pi)\ln\left(\frac{2}{1}\right)$$
$$= 240(\ln 2)\left[\cos(6 \times 10^8 \pi t - 0.2\pi) - \cos(6 \times 10^8 \pi t)\right]$$

10.4b (continued). Now for the right hand side. First,

$$\int \mathbf{B} \cdot d\mathbf{S} = \int_0^{0.1} \int_{.01}^{.02} \frac{8\pi \times 10^{-7}}{\rho} \cos(6 \times 10^8 \pi t - 2\pi z) \, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, d\rho \, dz$$

$$= \int_0^{0.1} (8\pi \times 10^{-7}) \ln 2 \, \cos(6 \times 10^8 \pi t - 2\pi z) \, dz$$

$$= -4 \times 10^{-7} \ln 2 \left[\sin(6 \times 10^8 \pi t - 0.2\pi) - \sin(6 \times 10^8 \pi t) \right]$$

Then

$$-\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = 240\pi (\ln 2) \left[\cos(6 \times 10^8 \pi t - 0.2\pi) - \cos(6 \times 10^8 \pi t) \right] \quad \text{(check)}$$

10.5. The location of the sliding bar in Fig. 10.5 is given by $x = 5t + 2t^3$, and the separation of the two rails is 20 cm. Let $\mathbf{B} = 0.8x^2 \mathbf{a}_z$ T. Find the voltmeter reading at:

a) $t = 0.4$ s: The flux through the loop will be

$$\Phi = \int_0^{0.2} \int_0^x 0.8(x')^2 \, dx' \, dy = \frac{0.16}{3} x^3 = \frac{0.16}{3} (5t + 2t^3)^3 \text{ Wb}$$

Then

$$\text{emf} = -\frac{d\Phi}{dt} = \frac{0.16}{3}(3)(5t + 2t^3)^2(5 + 6t^2) = -(0.16)[5(.4) + 2(.4)^3]^2[5 + 6(.4)^2] = \underline{-4.32 \text{ V}}$$

b) $x = 0.6$ m: Have $0.6 = 5t + 2t^3$, from which we find $t = 0.1193$. Thus

$$\text{emf} = -(0.16)[5(.1193) + 2(.1193)^3]^2[5 + 6(.1193)^2] = \underline{-.293 \text{ V}}$$

10.6. A perfectly conducting filament containing a small 500-Ω resistor is formed into a square, as illustrated in Fig. 10.6. Find $I(t)$ if

a) $\mathbf{B} = 0.3 \cos(120\pi t - 30°) \, \mathbf{a}_z$ T: First the flux through the loop is evaluated, where the unit normal to the loop is $\mathbf{a}_z$. We find

$$\Phi = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{S} = (0.3)(0.5)^2 \cos(120\pi t - 30°) \text{ Wb}$$

Then the current will be

$$I(t) = \frac{\text{emf}}{R} = -\frac{1}{R}\frac{d\Phi}{dt} = \frac{(120\pi)(0.3)(0.25)}{500} \sin(120\pi t - 30°) = \underline{57 \sin(120\pi t - 30°) \text{ mA}}$$

b) $\mathbf{B} = 0.4\cos[\pi(ct - y)]\,\mathbf{a}_z$ μT where $c = 3 \times 10^8$ m/s: Since the field varies with y, the flux is now

$$\Phi = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{S} = (0.5)(0.4)\int_0^{.5}\cos(\pi y - \pi ct)\,dy = \frac{0.2}{\pi}\left[\sin(\pi ct - \pi/2) - \sin(\pi ct)\right]\ \mu\text{Wb}$$

The current is then

$$\begin{aligned}
I(t) &= \frac{\text{emf}}{R} = -\frac{1}{R}\frac{d\Phi}{dt} = \frac{-0.2c}{500}\left[\cos(\pi ct - \pi/2) - \cos(\pi ct)\right]\ \mu\text{A} \\
&= \frac{-0.2(3 \times 10^8)}{500}\left[\sin(\pi ct) - \cos(\pi ct)\right]\ \mu\text{A} = \underline{120\left[\cos(\pi ct) - \sin(\pi ct)\right]\ \text{mA}}
\end{aligned}$$

10.7. The rails in Fig. 10.7 each have a resistance of 2.2 Ω/m. The bar moves to the right at a constant speed of 9 m/s in a uniform magnetic field of 0.8 T. Find $I(t)$, $0 < t < 1$ s, if the bar is at $x = 2$ m at $t = 0$ and

a) a 0.3 Ω resistor is present across the left end with the right end open-circuited: The flux in the left-hand closed loop is

$$\Phi_l = B \times \text{area} = (0.8)(0.2)(2 + 9t)$$

Then, emf$_l$ $= -d\Phi_l/dt = -(0.16)(9) = -1.44$ V. With the bar in motion, the loop resistance is increasing with time, and is given by $R_l(t) = 0.3 + 2[2.2(2 + 9t)]$. The current is now

$$I_l(t) = \frac{\text{emf}_l}{R_l(t)} = \frac{-1.44}{\underline{9.1 + 39.6t}}\ \text{A}$$

Note that the sign of the current indicates that it is flowing in the direction opposite that shown in the figure.

b) Repeat part a, but with a resistor of 0.3 Ω across each end: In this case, there will be a contribution to the current from the right loop, which is now closed. The flux in the right loop, whose area decreases with time, is

$$\Phi_r = (0.8)(0.2)[(16 - 2) - 9t]$$

and emf$_r$ $= -d\Phi_r/dt = (0.16)(9) = 1.44$ V. The resistance of the right loop is $R_r(t) = 0.3 + 2[2.2(14 - 9t)]$, and so the contribution to the current from the right loop will be

$$I_r(t) = \frac{-1.44}{61.9 - 39.6t}\ \text{A}$$

10.7b (continued). The minus sign has been inserted because again the current must flow in the opposite direction as that indicated in the figure, with the flux decreasing with time. The total current is found by adding the part a result, or

$$I_T(t) = -1.44 \left[\frac{1}{61.9 - 39.6t} + \frac{1}{9.1 + 39.6t} \right] \text{ A}$$

10.8. Fig. 10.1 is modified to show that the rail separation is larger when y is larger. Specifically, let the separation $d = 0.2 + 0.02y$. Given a uniform velocity $v_y = 8$ m/s and a uniform magnetic flux density $B_z = 1.1$ T, find V_{12} as a function of time if the bar is located at $y = 0$ at $t = 0$: The flux through the loop as a function of y can be written as

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} = \int_0^y \int_0^{.2+.02y'} 1.1 \, dx \, dy' = \int_0^y 1.1(.2 + .02y') \, dy' = 0.22y(1 + .05y)$$

Now, with $y = vt = 8t$, the above becomes $\Phi = 1.76t(1 + .40t)$. Finally,

$$V_{12} = -\frac{d\Phi}{dt} = -1.76(1 + .80t) \text{ V}$$

10.9. A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 Ω per meter length. The loop lies in the $z = 0$ plane with its corners at $(0,0,0)$, $(0.25, 0, 0)$, $(0.25, 0.25, 0)$, and $(0, 0.25, 0)$ at $t = 0$. The loop is moving with velocity $v_y = 50$ m/s in the field $B_z = 8\cos(1.5 \times 10^8 t - 0.5x)$ μT. Develop a function of time which expresses the ohmic power being delivered to the loop: First, since the field does not vary with y, the loop motion in the y direction does not produce any time-varying flux, and so this motion is immaterial. We can evaluate the flux at the original loop position to obtain:

$$\Phi(t) = \int_0^{.25} \int_0^{.25} 8 \times 10^{-6} \cos(1.5 \times 10^8 t - 0.5x) \, dx \, dy$$
$$= -(4 \times 10^{-6}) \left[\sin(1.5 \times 10^8 t - 0.13x) - \sin(1.5 \times 10^8 t) \right] \text{ Wb}$$

Now, $emf = V(t) = -d\Phi/dt = 6.0 \times 10^2 \left[\cos(1.5 \times 10^8 t - 0.13x) - \cos(1.5 \times 10^8 t) \right]$, The total loop resistance is $R = 125(0.25 + 0.25 + 0.25 + 0.25) = 125 \, \Omega$. Then the ohmic power is

$$P(t) = \frac{V^2(t)}{R} = 2.9 \times 10^3 \left[\cos(1.5 \times 10^8 t - 0.13x) - \cos(1.5 \times 10^8 t) \right] \text{ Watts}$$

10.10a. Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $E = E_m \cos\omega t$. Assume $\mu = \mu_0$. First, $D = \epsilon E = \epsilon E_m \cos\omega t$. Then the displacement current density is $\partial D/\partial t = -\omega\epsilon E_m \sin\omega t$. Second, $J_c = \sigma E = \sigma E_m \cos\omega t$. Using these results we find $|J_c|/|J_d| = \sigma/\omega\epsilon$.

b. What is the amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$, where τ is real? As before, find $D = \epsilon E = \epsilon E_m e^{-t/\tau}$, and so $J_d = \partial D/\partial t = -(\epsilon/\tau)E_m e^{-t/\tau}$. Also, $J_c = \sigma E_m e^{-t/\tau}$. Finally, $|J_c|/|J_d| = \underline{\sigma\tau/\epsilon}$.

10.11. Let the internal dimension of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The homogeneous material inside the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. If the electric field intensity is $\mathbf{E} = (10^6/\rho) \cos(10^5 t)\mathbf{a}_\rho$ V/m, find:

a) $\mathbf{J}$: Use
$$\mathbf{J} = \sigma\mathbf{E} = \left(\frac{10}{\rho}\right) \cos(10^5 t)\mathbf{a}_\rho \ \text{A/m}^2$$

b) the total conduction current, I_c, through the capacitor: Have
$$I_c = \int\int \mathbf{J} \cdot d\mathbf{S} = 2\pi\rho l J = 20\pi l \cos(10^5 t) = \underline{8\pi \cos(10^5 t)} \ \text{A}$$

c) the total displacement current, I_d, through the capacitor: First find
$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon\mathbf{E}) = -\frac{(10^5)(10^{-11})(10^6)}{\rho} \sin(10^5 t)\mathbf{a}_\rho = -\frac{1}{\rho} \sin(10^5 t) \ \text{A/m}$$

Now
$$I_d = 2\pi\rho l J_d = -2\pi l \sin(10^5 t) = \underline{-0.8\pi \sin(10^5 t)} \ \text{A}$$

d) the ratio of the amplitude of I_d to that of I_c, the quality factor of the capacitor: This will be
$$\frac{|I_d|}{|I_c|} = \frac{0.8}{8} = \underline{0.1}$$

10.12. Show that the displacement current flowing between the two conducting cylinders in a lossless coaxial capacitor is exactly the same as the conduction current flowing in the external circuit if the applied voltage between conductors is $V_0 \cos \omega t$ volts.

From Chapter 7, we know that for a given applied voltage between the cylinders, the electric field is

$$\mathbf{E} = \frac{V_0 \cos \omega t}{\rho \ln(b/a)} \mathbf{a}_\rho \text{ V/m} \quad \Rightarrow \quad \mathbf{D} = \frac{\epsilon V_0 \cos \omega t}{\rho \ln(b/a)} \mathbf{a}_\rho \text{ C/m}^2$$

Then the displacement current density is

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{-\omega \epsilon V_0 \sin \omega t}{\rho \ln(b/a)} \mathbf{a}_\rho$$

Over a length ℓ, the displacement current will be

$$I_d = \int\int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} = 2\pi \rho \ell \frac{\partial \mathbf{D}}{\partial t} = \frac{2\pi \ell \omega \epsilon V_0 \sin \omega t}{\ln(b/a)} = C\frac{dV}{dt} = I_c$$

where we recall that the capacitance is given by $C = 2\pi \epsilon \ell / \ln(b/a)$.

10.13. Consider the region defined by $|x|$, $|y|$, and $|z| < 1$. Let $\epsilon_r = 5$, $\mu_r = 4$, and $\sigma = 0$. If $\mathbf{J}_d = 20 \cos(1.5 \times 10^8 t - bx)\mathbf{a}_y \ \mu\text{A/m}^2$;

a) find $\mathbf{D}$ and $\mathbf{E}$: Since $\mathbf{J}_d = \partial \mathbf{D}/\partial t$, we write

$$\mathbf{D} = \int \mathbf{J}_d dt + C = \frac{20 \times 10^{-6}}{1.5 \times 10^8} \sin(1.5 \times 10^8 - bx)\mathbf{a}_y$$
$$= \underline{1.33 \times 10^{-13} \sin(1.5 \times 10^8 t - bx)\mathbf{a}_y \text{ C/m}^2}$$

where the integration constant is set to zero (assuming no dc fields are present). Then

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{1.33 \times 10^{-13}}{(5 \times 8.85 \times 10^{-12})} \sin(1.5 \times 10^8 t - bx)\mathbf{a}_y$$
$$= \underline{3.0 \times 10^{-3} \sin(1.5 \times 10^8 t - bx)\mathbf{a}_y \text{ V/m}}$$

b) use the point form of Faraday's law and an integration with respect to time to find $\mathbf{B}$ and $\mathbf{H}$: In this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = -b(3.0 \times 10^{-3}) \cos(1.5 \times 10^8 t - bx)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

Solve for $\mathbf{B}$ by integrating over time:

$$\mathbf{B} = \frac{b(3.0 \times 10^{-3})}{1.5 \times 10^8} \sin(1.5 \times 10^8 t - bx)\mathbf{a}_z = \underline{(2.0)b \times 10^{-11} \sin(1.5 \times 10^8 t - bx)\mathbf{a}_z \text{ T}}$$

10.13b (continued). Now

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{(2.0)b \times 10^{-11}}{4 \times 4\pi \times 10^{-7}} \sin(1.5 \times 10^8 t - bx)\mathbf{a}_z$$
$$= \underline{(4.0 \times 10^{-6})b \sin(1.5 \times 10^8 t - bx)\mathbf{a}_z \text{ A/m}}$$

c) use $\nabla \times \mathbf{H} = \mathbf{J}_d + \mathbf{J}$ to find $\mathbf{J}_d$: Since $\sigma = 0$, there is no conduction current, so in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{a}_y = \underline{4.0 \times 10^{-6}b^2 \cos(1.5 \times 10^8 t - bx)\mathbf{a}_y \text{ A/m}^2} = \mathbf{J}_d$$

d) What is the numerical value of b? We set the given expression for $\mathbf{J}_d$ equal to the result of part c to obtain:

$$20 \times 10^{-6} = 4.0 \times 10^{-6}b^2 \quad \Rightarrow \quad b = \underline{\sqrt{5.0} \text{ m}^{-1}}$$

10.14. A voltage source, $V_0 \sin \omega t$, is connected between two concentric conducting spheres, $r = a$ and $r = b$, $b > a$, where the region between them is a material for which $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. Find the total displacement current through the dielectric and compare it with the source current as determined from the capacitance (Sec. 5.10) and circuit analysis methods: First, solving Laplace's equation, we find the voltage between spheres (see Eq. 20, Chapter 7):

$$V(t) = \frac{(1/r) - (1/b)}{(1/a) - (1/b)} V_0 \sin \omega t$$

Then

$$\mathbf{E} = -\nabla V = \frac{V_0 \sin \omega t}{r^2(1/a - 1/b)}\mathbf{a}_r \quad \Rightarrow \quad \mathbf{D} = \frac{\epsilon_r \epsilon_0 V_0 \sin \omega t}{r^2(1/a - 1/b)}\mathbf{a}_r$$

Now

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t} = \frac{\epsilon_r \epsilon_0 \omega V_0 \cos \omega t}{r^2(1/a - 1/b)}\mathbf{a}_r$$

The displacement current is then

$$I_d = 4\pi r^2 J_d = \frac{4\pi \epsilon_r \epsilon_0 \omega V_0 \cos \omega t}{(1/a - 1/b)} = C\frac{dV}{dt}$$

where, from Eq. 47, Chapter 5,

$$C = \frac{4\pi \epsilon_r \epsilon_0}{(1/a - 1/b)}$$

The results are consistent.

10.15. Let $\mu = 3 \times 10^{-5}$ H/m, $\epsilon = 1.2 \times 10^{-10}$ F/m, and $\sigma = 0$ everywhere. If $\mathbf{H} = 2\cos(10^{10}t - \beta x)\mathbf{a}_z$ A/m, use Maxwell's equations to obtain expressions for $\mathbf{B}$, $\mathbf{D}$, $\mathbf{E}$, and β: First, $\mathbf{B} = \mu\mathbf{H} = \underline{6 \times 10^{-5}\cos(10^{10}t - \beta x)\mathbf{a}_z}$ T. Next we use

$$\nabla \times \mathbf{H} = -\frac{\partial \mathbf{H}}{\partial x}\mathbf{a}_y = 2\beta\sin(10^{10}t - \beta x)\mathbf{a}_y = \frac{\partial \mathbf{D}}{\partial t}$$

from which

$$\mathbf{D} = \int 2\beta\sin(10^{10}t - \beta x)\,dt + C = \underline{-\frac{2\beta}{10^{10}}\cos(10^{10}t - \beta x)\mathbf{a}_y} \text{ C/m}^2$$

where the integration constant is set to zero, since no dc fields are presumed to exist. Next,

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = -\frac{2\beta}{(1.2 \times 10^{-10})(10^{10})}\cos(10^{10}t - \beta x)\mathbf{a}_y = \underline{-1.67\beta\cos(10^{10}t - \beta x)\mathbf{a}_y} \text{ V/m}$$

Now

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = 1.67\beta^2\sin(10^{10}t - \beta x)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

So

$$\mathbf{B} = -\int 1.67\beta^2\sin(10^{10}t - \beta x)\mathbf{a}_z\,dt = (1.67 \times 10^{-10})\beta^2\cos(10^{10}t - \beta x)\mathbf{a}_z$$

We require this result to be consistent with the expression for $\mathbf{B}$ originally found. So

$$(1.67 \times 10^{-10})\beta^2 = 6 \times 10^{-5} \quad \Rightarrow \quad \beta = \pm 600 \text{ rad/m}$$

10.16. Derive the continuity equation from Maxwell's equations: First, take the divergence of both sides of Ampere's circuital law:

$$\underbrace{\nabla \cdot \nabla \times \mathbf{H}}_{0} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}\nabla \cdot \mathbf{D} = \underline{\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0}$$

where we have used $\nabla \cdot \mathbf{D} = \rho_v$, another Maxwell equation.

10.17. The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$, $0 < z < 0.06$ m in free space is given by $\mathbf{E} = C\sin(12y)\sin(az)\cos(2 \times 10^{10}t)\,\mathbf{a}_x$ V/m. Beginning with the $\nabla \times \mathbf{E}$ relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than zero: In this case we find

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y - \frac{\partial E_x}{\partial y}\mathbf{a}_z$$

$$= C\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\cos(2 \times 10^{10}t) = -\frac{\partial \mathbf{B}}{\partial t}$$

10.17 (continued). Then

$$\mathbf{H} = -\frac{1}{\mu_0} \int \nabla \times \mathbf{E} \, dt + C_1$$

$$= -\frac{C}{\mu_0(2 \times 10^{10})} \left[a \sin(12y) \cos(az)\mathbf{a}_y - 12 \cos(12y) \sin(az)\mathbf{a}_z \right] \sin(2 \times 10^{10}t) \ \text{A/m}$$

where the integration constant, $C_1 = 0$, since there are no initial conditions. Using this result, we now find

$$\nabla \times \mathbf{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right] \mathbf{a}_x = -\frac{C(144 + a^2)}{\mu_0(2 \times 10^{10})} \sin(12y) \sin(az) \sin(2 \times 10^{10}t)\, \mathbf{a}_x = \frac{\partial \mathbf{D}}{\partial t}$$

Now

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \int \frac{1}{\epsilon_0} \nabla \times \mathbf{H} \, dt + C_2 = \frac{C(144 + a^2)}{\mu_0 \epsilon_0 (2 \times 10^{10})^2} \sin(12y) \sin(az) \cos(2 \times 10^{10}t)\, \mathbf{a}_x$$

where $C_2 = 0$. This field must be the same as the original field as stated, and so we require that

$$\frac{C(144 + a^2)}{\mu_0 \epsilon_0 (2 \times 10^{10})^2} = 1$$

Using $\mu_0 \epsilon_0 = (3 \times 10^8)^{-2}$, we find

$$a = \left[\frac{(2 \times 10^{10})^2}{(3 \times 10^8)^2} - 144 \right]^{1/2} = \underline{66 \ \text{m}^{-1}}$$

10.18. The parallel plate transmission line shown in Fig. 10.8 has dimensions $b = 4$ cm and $d = 8$ mm, while the medium between plates is characterized by $\mu_r = 1$, $\epsilon_r = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5\cos(10^9 t - \beta z)\mathbf{a}_y$ A/m, use Maxwell's equations to help find:

a) β, if $\beta > 0$: Take

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = -5\beta \sin(10^9 t - \beta z)\mathbf{a}_x = 20\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

So

$$\mathbf{E} = \int \frac{-5\beta}{20\epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_x \, dt = \frac{\beta}{(4 \times 10^9)\epsilon_0} \cos(10^9 t - \beta z)\mathbf{a}_x$$

Then

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = \frac{\beta^2}{(4 \times 10^9)\epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_y = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

So that

$$\mathbf{H} = \int \frac{-\beta^2}{(4 \times 10^9)\mu_0 \epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_x \, dt = \frac{\beta^2}{(4 \times 10^{18})\mu_0 \epsilon_0} \cos(10^9 t - \beta z)$$

$$= 5\cos(10^9 t - \beta z)\mathbf{a}_y$$

10.18a (continued) where the last equality is required to maintain consistency. Therefore

$$\frac{\beta^2}{(4 \times 10^{18})\mu_0\epsilon_0} = 5 \quad \Rightarrow \quad \beta = \underline{14.9 \text{ m}^{-1}}$$

b) the displacement current density at $z = 0$: Since $\sigma = 0$, we have

$$\nabla \times \mathbf{H} = \mathbf{J}_d = -5\beta \sin(10^9 t - \beta z) = -74.5 \sin(10^9 t - 14.9z)\mathbf{a}_x$$

$$= \underline{-74.5 \sin(10^9 t)\mathbf{a}_x \text{ A/m}} \text{ at } z = 0$$

c) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, and $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction. We evaluate the flux integral of $\mathbf{J}_d$ over the given cross section:

$$I_d = -74.5b \int_0^{0.1} \sin(10^9 t - 14.9z)\, \mathbf{a}_x \cdot \mathbf{a}_x\, dz = \underline{0.20\left[\cos(10^9 t - 1.49) - \cos(10^9 t)\right] \text{ A}}$$

10.19. In the first section of this chapter, Faraday's law was used to show that the field $\mathbf{E} = -\frac{1}{2}kB_0\rho e^{kt}\mathbf{a}_\phi$ results from the changing magnetic field $\mathbf{B} = B_0 e^{kt}\mathbf{a}_z$.

a) Show that these fields do not satisfy Maxwell's other curl equation: Note that $\mathbf{B}$ as stated is constant with position, and so will have zero curl. The electric field, however, varies with time, and so $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$ would have a zero left-hand side and a non-zero right-hand side. The equation is thus not valid with these fields.

b) If we let $B_0 = 1$ T and $k = 10^6$ s^{-1}, we are establishing a fairly large magnetic flux density in 1 μs. Use the $\nabla \times \mathbf{H}$ equation to show that the rate at which B_z should (but does not) change with ρ is only about 5×10^{-6} T/m in free space at $t = 0$: Assuming that $\mathbf{B}$ varies with ρ, we write

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial \rho}\mathbf{a}_\phi = -\frac{1}{\mu_0}\frac{dB_0}{d\rho}e^{kt} = \epsilon_0\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{2}\epsilon_0 k^2 B_0 \rho e^{kt}$$

Thus

$$\frac{dB_0}{d\rho} = \frac{1}{2}\mu_0\epsilon_0 k^2 \rho B_0 = \frac{10^{12}(1)\rho}{2(3 \times 10^8)^2} = 5.6 \times 10^{-6}\rho$$

which is near the stated value if ρ is on the order of 1m.

10.20. Point $C(-0.1, -0.2, 0.3)$ lies on the surface of a perfect conductor. The electric field intensity at C is $(500\mathbf{a}_x - 300\mathbf{a}_y + 600\mathbf{a}_z) \cos 10^7 t$ V/m, and the medium surrounding the conductor is characterized by $\mu_r = 5$, $\epsilon_r = 10$, and $\sigma = 0$.

a) Find a unit vector normal to the conductor surface at C, if the origin lies within the conductor: At $t = 0$, the field must be directed *out of* the surface, and will be normal to it, since we have a perfect conductor. Therefore

$$\mathbf{n} = \frac{+\mathbf{E}(t = 0)}{|\mathbf{E}(t = 0)|} = \frac{5\mathbf{a}_x - 3\mathbf{a}_y + 6\mathbf{a}_z}{\sqrt{25 + 9 + 36}} = \underline{0.60\mathbf{a}_x - 0.36\mathbf{a}_y + 0.72\mathbf{a}_z}$$

b) Find the surface charge density at C: Use

$$\rho_s = \mathbf{D} \cdot \mathbf{n}|_{surface} = 10\epsilon_0 \left[500\mathbf{a}_x - 300\mathbf{a}_y + 600\mathbf{a}_z\right] \cos(10^7 t) \cdot \left[.60\mathbf{a}_x - .36\mathbf{a}_y + .72\mathbf{a}_z\right]$$

$$= 10\epsilon_0 \left[300 + 108 + 432\right] \cos(10^7 t) = 7.4 \times 10^{-8} \cos(10^7 t) \text{ C/m}^2$$

$$= \underline{74 \cos(10^7 t) \text{ nC/m}^2}$$

10.21. a) Show that under static field conditions, Eq. (55) reduces to Ampere's circuital law. First use the definition of the vector Laplacian:

$$\nabla^2 \mathbf{A} = -\nabla \times \nabla \times \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\mu\mathbf{J}$$

which is Eq. (55) with the time derivative set to zero. We also note that $\nabla \cdot \mathbf{A} = 0$ in steady state (from Eq. (54)). Now, since $\mathbf{B} = \nabla \times \mathbf{A}$, (55) becomes

$$-\nabla \times \mathbf{B} = -\mu\mathbf{J} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}$$

b) Show that Eq. (51) becomes Faraday's law when taking the curl: Doing this gives

$$\nabla \times \mathbf{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t}\nabla \times \mathbf{A}$$

The curl of the gradient is identically zero, and $\nabla \times \mathbf{A} = \mathbf{B}$. We are left with

$$\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$$

10.22. In a sourceless medium, in which $\mathbf{J} = 0$ and $\rho_v = 0$, assume a rectangular coordinate system in which $\mathbf{E}$ and $\mathbf{H}$ are functions only of z and t. The medium has permittivity ϵ and permeability μ. (a) If $\mathbf{E} = E_x \mathbf{a}_x$ and $\mathbf{H} = H_y \mathbf{a}_y$, begin with Maxwell's equations and determine the second order partial differential equation that E_x must satisfy.

First use

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \frac{\partial E_x}{\partial z} \mathbf{a}_y = -\mu \frac{\partial H_y}{\partial t} \mathbf{a}_y$$

in which case, the curl has dictated the direction that $\mathbf{H}$ must lie in. Similarly, use the other Maxwell curl equation to find

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad \Rightarrow \quad -\frac{\partial H_y}{\partial z} \mathbf{a}_x = \epsilon \frac{\partial E_x}{\partial t} \mathbf{a}_x$$

Now, differentiate the first equation with respect to z, and the second equation with respect to t:

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu \frac{\partial^2 H_y}{\partial t \partial z} \quad \text{and} \quad \frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

Combining these two, we find

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \epsilon \frac{\partial^2 E_x}{\partial t^2}$$

b) Show that $E_x = E_0 \cos(\omega t - \beta z)$ is a solution of that equation for a particular value of β: Substituting, we find

$$\frac{\partial^2 E_x}{\partial z^2} = -\beta^2 E_0 \cos(\omega t - \beta z) \quad \text{and} \quad \mu \epsilon \frac{\partial^2 E_x}{\partial t^2} = -\omega^2 \mu \epsilon E_0 \cos(\omega t - \beta z)$$

These two will be equal provided the constant multipliers of $\cos(\omega t - \beta z)$ are equal.

c) Find β as a function of given parameters. Equating the two constants in part b, we find

$\underline{\beta = \omega \sqrt{\mu \epsilon}}$.

10.23. In region 1, $z < 0$, $\epsilon_1 = 2 \times 10^{-11}$ F/m, $\mu_1 = 2 \times 10^{-6}$ H/m, and $\sigma_1 = 4 \times 10^{-3}$ S/m; in region 2, $z > 0$, $\epsilon_2 = \epsilon_1/2$, $\mu_2 = 2\mu_1$, and $\sigma_2 = \sigma_1/4$. It is known that $\mathbf{E}_1 = (30\mathbf{a}_x + 20\mathbf{a}_y + 10\mathbf{a}_z) \cos(10^9 t)$ V/m at $P_1(0, 0, 0^-)$.

a) Find $\mathbf{E}_{N1}$, $\mathbf{E}_{t1}$, $\mathbf{D}_{N1}$, and $\mathbf{D}_{t1}$: These will be

$$\mathbf{E}_{N1} = \underline{10 \cos(10^9 t) \mathbf{a}_z \text{ V/m}} \quad \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y) \cos(10^9 t) \text{ V/m}}$$

$$\mathbf{D}_{N1} = \epsilon_1 \mathbf{E}_{N1} = (2 \times 10^{-11})(10) \cos(10^9 t) \mathbf{a}_z \text{ C/m}^2 = \underline{200 \cos(10^9 t) \mathbf{a}_z \text{ pC/m}^2}$$

10.23a (continued).

$$\mathbf{D}_{t1} = \epsilon_1 \mathbf{E}_{t1} = (2 \times 10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(600\mathbf{a}_x + 400\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

b) Find $\mathbf{J}_{N1}$ and $\mathbf{J}_{t1}$ at P_1:

$$\mathbf{J}_{N1} = \sigma_1 \mathbf{E}_{N1} = (4 \times 10^{-3})(10\cos(10^9 t))\mathbf{a}_z = \underline{40\cos(10^9 t)\mathbf{a}_z \text{ mA/m}^2}$$

$$\mathbf{J}_{t1} = \sigma_1 \mathbf{E}_{t1} = (4 \times 10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(120\mathbf{a}_x + 80\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

c) Find $\mathbf{E}_{t2}$, $\mathbf{D}_{t2}$, and $\mathbf{J}_{t2}$ at P_1: By continuity of tangential $\mathbf{E}$,

$$\mathbf{E}_{t2} = \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ V/m}}$$

Then

$$\mathbf{D}_{t2} = \epsilon_2 \mathbf{E}_{t2} = (10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(300\mathbf{a}_x + 200\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

$$\mathbf{J}_{t2} = \sigma_2 \mathbf{E}_{t2} = (10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

d) (Harder) Use the continuity equation to help show that $J_{N1} - J_{N2} = \partial D_{N2}/\partial t - \partial D_{N1}/\partial t$ and then determine $\mathbf{E}_{N2}$, $\mathbf{D}_{N2}$, and $\mathbf{J}_{N2}$: We assume the existence of a surface charge layer at the boundary having density ρ_s C/m^2. If we draw a cylindrical "pillbox" whose top and bottom surfaces (each of area Δa) are on either side of the interface, we may use the continuity condition to write

$$(J_{N2} - J_{N1})\Delta a = -\frac{\partial \rho_s}{\partial t}\Delta a$$

where $\rho_s = D_{N2} - D_{N1}$. Therefore,

$$J_{N1} - J_{N2} = \frac{\partial}{\partial t}(D_{N2} - D_{N1})$$

In terms of the normal electric field components, this becomes

$$\sigma_1 E_{N1} - \sigma_2 E_{N2} = \frac{\partial}{\partial t}(\epsilon_2 E_{N2} - \epsilon_1 E_{N1})$$

Now let $E_{N2} = A\cos(10^9 t) + B\sin(10^9 t)$, while from before, $E_{N1} = 10\cos(10^9 t)$.

These, along with the permittivities and conductivities, are substituted to obtain

$$(4 \times 10^{-3})(10)\cos(10^9 t) - 10^{-3}[A\cos(10^9 t) + B\sin(10^9 t)]$$
$$= \frac{\partial}{\partial t}\left[10^{-11}[A\cos(10^9 t) + B\sin(10^9 t)] - (2 \times 10^{-11})(10)\cos(10^9 t)\right]$$
$$= -(10^{-2}A\sin(10^9 t) + 10^{-2}B\cos(10^9 t) + (2 \times 10^{-1})\sin(10^9 t)$$

We now equate coefficients of the sin and cos terms to obtain two equations:

$$4 \times 10^{-2} - 10^{-3}A = 10^{-2}B$$

$$-10^{-3}B = -10^{-2}A + 2 \times 10^{-1}$$

These are solved together to find $A = 20.2$ and $B = 2.0$. Thus

$$\mathbf{E}_{N2} = \left[20.2\cos(10^9 t) + 2.0\sin(10^9 t)\right]\mathbf{a}_z = \underline{20.3\cos(10^9 t + 5.6°)\mathbf{a}_z \text{ V/m}}$$

Then

$$\mathbf{D}_{N2} = \epsilon_2\mathbf{E}_{N2} = \underline{203\cos(10^9 t + 5.6°)\mathbf{a}_z \text{ pC/m}^2}$$

and

$$\mathbf{J}_{N2} = \sigma_2\mathbf{E}_{N2} = \underline{20.3\cos(10^9 t + 5.6°)\mathbf{a}_z \text{ mA/m}^2}$$

10.24. In a medium in which $\rho_v = 0$, but in which the permittivity is a function of position, determine the conditions on the permittivity variation such that

a) $\nabla \cdot \mathbf{E} = 0$: We first note that $\nabla \cdot \mathbf{D} = 0$ if $\rho_v = 0$, where $\mathbf{D} = \epsilon\mathbf{E}$. Now

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon\mathbf{E}) = \mathbf{E} \cdot \nabla\epsilon + \epsilon\nabla \cdot \mathbf{E} = 0$$

or

$$\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\nabla\epsilon}{\epsilon} = 0$$

We see that $\nabla \cdot \mathbf{E} = 0$ if $\nabla\epsilon = 0$.

b) $\nabla \cdot \mathbf{E} \doteq 0$: From the development in part a, $\nabla \cdot \mathbf{E}$ will be approximately zero if $\nabla\epsilon/\epsilon$ is negligible.

10.25. In a region where $\mu_r = \epsilon_r = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ V and $\mathbf{A} = x[(z/c) - t]\mathbf{a}_z$ Wb/m, where $c = 1/\sqrt{\mu_0\epsilon_0}$.

a) Show that $\nabla \cdot \mathbf{A} = -\mu\epsilon(\partial V/\partial t)$:

First,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{x}{c} = x\sqrt{\mu_0\epsilon_0}$$

Second,

$$\frac{\partial V}{\partial t} = -cx = -\frac{x}{\sqrt{\mu_0\epsilon_0}}$$

so we observe that $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0(\partial V/\partial t)$ in free space, implying that the given statement would hold true in general media.

b) Find $\mathbf{B}$, $\mathbf{H}$, $\mathbf{E}$, and $\mathbf{D}$:

Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_x}{\partial x}\mathbf{a}_y = \underline{\left(t - \frac{z}{c}\right)\mathbf{a}_y} \text{ T}$$

Then

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \underline{\frac{1}{\mu_0}\left(t - \frac{z}{c}\right)\mathbf{a}_y} \text{ A/m}$$

Now,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -(z - ct)\mathbf{a}_x - x\mathbf{a}_z + x\mathbf{a}_z = \underline{(ct - z)\mathbf{a}_x} \text{ V/m}$$

Then

$$\mathbf{D} = \epsilon_0\mathbf{E} = \underline{\epsilon_0(ct - z)\mathbf{a}_x} \text{ C/m}^2$$

c) Show that these results satisfy Maxwell's equations if $\mathbf{J}$ and ρ_v are zero:

i. $\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon_0(ct - z)\mathbf{a}_x = 0$

ii. $\nabla \cdot \mathbf{B} = \nabla \cdot (t - z/c)\mathbf{a}_y = 0$

iii.

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = \frac{1}{\mu_0 c}\mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

which we require to equal $\partial \mathbf{D}/\partial t$:

$$\frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 c\mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

10.25c (continued).

iv.

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\mathbf{a}_y$$

which we require to equal $-\partial \mathbf{B}/\partial t$:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{a}_y$$

So all four Maxwell equations are satisfied.

10.26. Let the current $I = 80t$ A be present in the $\mathbf{a}_z$ direction on the z axis in free space within the interval $-0.1 < z < 0.1$ m.

a) Find A_z at $P(0,2,0)$: The integral for the retarded vector potential will in this case assume the form

$$\mathbf{A} = \int_{-.1}^{.1} \frac{\mu_0 80(t - R/c)}{4\pi R}\mathbf{a}_z \, dz$$

where $R = \sqrt{z^2 + 4}$ and $c = 3 \times 10^8$ m/s. We obtain

$$A_z = \frac{80\mu_0}{4\pi}\left[\int_{-.1}^{.1} \frac{t}{\sqrt{z^2 + 4}}\, dz - \int_{-.1}^{.1}\frac{1}{c}\, dz\right] = 8 \times 10^{-6}t \ln(z + \sqrt{z^2 + 4})\Big|_{-.1}^{.1} - \frac{8 \times 10^{-6}}{3 \times 10^8}z\Big|_{-.1}^{.1}$$

$$= 8 \times 10^{-6} \ln\left(\frac{.1 + \sqrt{4.01}}{-.1 + \sqrt{4.01}}\right) - 0.53 \times 10^{-14} = 8.0 \times 10^{-7}t - 0.53 \times 10^{-14}$$

So finally, $\mathbf{A} = \underline{\left[8.0 \times 10^{-7}t - 5.3 \times 10^{-15}\right]\mathbf{a}_z}$ Wb/m.

b) Sketch A_z versus t over the time interval $-0.1 < t < 0.1$ μs: The sketch is linearly increasing with time, beginning with $A_z = -8.53 \times 10^{-14}$ Wb/m at $t = -0.1$ μs, crossing the time axis and going positive at $t = 6.6$ ns, and reaching a maximum value of 7.46×10^{-14} Wb/m at $t = 0.1$ μs.