CHAPTER 13

13.1. A uniform plane wave in air, $E_{x1}^+ = E_{x10}^+ \cos(10^{10}t - \beta z)$ V/m, is normally-incident on a copper surface at $z = 0$. What percentage of the incident power density is transmitted into the copper? We need to find the reflection coefficient. The intrinsic impedance of copper (a good conductor) is

$$\eta_c = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\sqrt{\frac{10^{10}(4\pi \times 10^7)}{2(5.8 \times 10^7)}} = (1+j)(.0104)$$

Note that the accuracy here is questionable, since we know the conductivity to only two significant figures. We nevertheless proceed: Using $\eta_0 = 376.7288$ ohms, we write

$$\Gamma = \frac{\eta_c - \eta_0}{\eta_c + \eta_0} = \frac{.0104 - 376.7288 + j.0104}{.0104 + 376.7288 + j.0104} = -.9999 + j.0001$$

Now $|\Gamma|^2 = .9999$, and so the transmitted power fraction is $1 - |\Gamma|^2 = .0001$, or about 0.01% is transmitted.

13.2. The plane $z = 0$ defines the boundary between two dielectrics. For $z < 0$, $\epsilon_{r1} = 5$, $\epsilon''_{r1} = 0$, and $\mu_1 = \mu_0$. For $z > 0$, $\epsilon'_{r2} = 3$, $\epsilon''_{r2} = 0$, and $\mu_2 = \mu_0$. Let $E_{x1}^+ = 200\cos(\omega t - 15z)$ V/m and find

a) ω: We have $\beta = \omega\sqrt{\mu_0\epsilon'_1} = \omega\sqrt{\epsilon'_{r1}}/c = 15$. So $\omega = 15c/\sqrt{\epsilon'_{r1}} = 15 \times (3 \times 10^8)/\sqrt{5} = \underline{2.0 \times 10^9 \text{ s}^{-1}}$.

b) $< \mathbf{S}_1^+ >$: First we need $\eta_1 = \sqrt{\mu_0/\epsilon'_1} = \eta_0/\sqrt{\epsilon'_{r1}} = 377/\sqrt{5} = 169$ ohms. Next we apply Eq. (76), Chapter 12, to evaluate the Poynting vector (with no loss and consequently with no phase difference between electric and magnetic fields). We find $< \mathbf{S}_1^+ > = (1/2)|E_1|^2/\eta_1\, \mathbf{a}_z = (1/2)(200)^2/169\, \mathbf{a}_z = \underline{119\, \mathbf{a}_z \text{ W/m}^2}$.

c) $< \mathbf{S}_1^- >$: First, we need to evaluate the reflection coefficient:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0/\sqrt{\epsilon'_{r2}} - \eta_0/\sqrt{\epsilon'_{r1}}}{\eta_0/\sqrt{\epsilon'_{r2}} + \eta_0/\sqrt{\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}} - \sqrt{\epsilon'_{r2}}}{\sqrt{\epsilon'_{r1}} + \sqrt{\epsilon'_{r2}}} = \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} = 0.13$$

Then $< \mathbf{S}_1^- > = -|\Gamma|^2 < \mathbf{S}_1^+ > = -(0.13)^2(119)\, \mathbf{a}_z = \underline{-2.0\, \mathbf{a}_z \text{ W/m}^2}$.

d) $< \mathbf{S}_2^+ >$: This will be the remaining power, propagating in the forward z direction, or $< \mathbf{S}_2^+ > = \underline{117\, \mathbf{a}_z \text{ W/m}^2}$.

13.3. A uniform plane wave in region 1 is normally-incident on the planar boundary separating regions 1 and 2. If $\epsilon''_1 = \epsilon''_2 = 0$, while $\epsilon'_{r1} = \mu_{r1}^3$ and $\epsilon'_{r2} = \mu_{r2}^3$, find the ratio $\epsilon'_{r2}/\epsilon'_{r1}$ if 20% of the energy in the incident wave is reflected at the boundary. There are two possible answers. First, since $|\Gamma|^2 = .20$, and since both permittivities and permeabilities are real, $\Gamma = \pm 0.447$. we then set up

$$\Gamma = \pm 0.447 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{(\mu_{r2}/\epsilon'_{r2})} - \eta_0\sqrt{(\mu_{r1}/\epsilon'_{r1})}}{\eta_0\sqrt{(\mu_{r2}/\epsilon'_{r2})} + \eta_0\sqrt{(\mu_{r1}/\epsilon'_{r1})}}$$

$$= \frac{\sqrt{(\mu_{r2}/\mu_{r2}^3)} - \sqrt{(\mu_{r1}/\mu_{r1}^3)}}{\sqrt{(\mu_{r2}/\mu_{r2}^3)} + \sqrt{(\mu_{r1}/\mu_{r1}^3)}} = \frac{\mu_{r1} - \mu_{r2}}{\mu_{r1} + \mu_{r2}}$$

13.3. (continued) Therefore

$$\frac{\mu_{r2}}{\mu_{r1}} = \frac{1 \mp 0.447}{1 \pm 0.447} = (0.382, 2.62) \quad \Rightarrow \quad \frac{\epsilon'_{r2}}{\epsilon'_{r1}} = \left(\frac{\mu_{r2}}{\mu_{r1}}\right)^3 = \underline{(0.056, 17.9)}$$

13.4. A 10-MHz uniform plane wave having an initial average power density of 5W/m^2 is normally-incident from free space onto the surface of a lossy material in which $\epsilon''_2/\epsilon'_2 = 0.05$, $\epsilon'_{r2} = 5$, and $\mu_2 = \mu_0$. Calculate the distance into the lossy medium at which the transmitted wave power density is down by 10dB from the initial 5W/m^2:

First, since $\epsilon''_2/\epsilon'_2 = 0.05 << 1$, we recognize region 2 as a good dielectric. Its intrinsic impedance is therefore approximated well by Eq. (62b), Chapter 12:

$$\eta_2 = \sqrt{\frac{\mu_0}{\epsilon'_2}}\left[1 + j\frac{1}{2}\frac{\epsilon''_2}{\epsilon'_2}\right] = \frac{377}{\sqrt{5}}\left[1 + j0.025\right]$$

The reflection coefficient encountered by the incident wave from region 1 is therefore

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(377/\sqrt{5})[1 + j.025] - 377}{(377/\sqrt{5})[1 + j.025] + 377} = \frac{(1 - \sqrt{5}) + j.025}{(1 + \sqrt{5}) + j.025} = -0.383 + j0.011$$

The fraction of the incident power that is reflected is then $|\Gamma|^2 = 0.147$, and thus the fraction of the power that is transmitted into region 2 is $1 - |\Gamma|^2 = 0.853$. Still using the good dielectric approximation, the attenuation coefficient in region 2 is found from Eq. (60a), Chapter 12:

$$\alpha \doteq \frac{\omega\epsilon''_2}{2}\sqrt{\frac{\mu_0}{\epsilon'_2}} = (2\pi \times 10^7)(0.05 \times 5 \times 8.854 \times 10^{-12})\frac{377}{2\sqrt{5}} = 2.34 \times 10^{-2} \text{ Np/m}$$

Now, the power that propagates into region 2 is expressed in terms of the incident power through

$$< S_2 > (z) = 5(1 - |\Gamma|^2)e^{-2\alpha z} = 5(.853)e^{-2(2.34\times10^{-2})z} = 0.5 \text{ W/m}^2$$

in which the last equality indicates a factor of ten reduction from the incident power, as occurs for a 10 dB loss. Solve for z to obtain

$$z = \frac{\ln(8.53)}{2(2.34 \times 10^{-2})} = \underline{45.8 \text{ m}}$$

13.5. The region $z < 0$ is characterized by $\epsilon'_r = \mu_r = 1$ and $\epsilon''_r = 0$. The total $\mathbf{E}$ field here is given as the sum of the two uniform plane waves, $\mathbf{E}_s = 150e^{-j10z}\,\mathbf{a}_x + (50\angle 20^\circ)e^{j10z}\,\mathbf{a}_x$ V/m.

a) What is the operating frequency? In free space, $\beta = k_0 = 10 = \omega/c = \omega/3 \times 10^8$. Thus, $\omega = 3 \times 10^9 \text{ s}^{-1}$, or $f = \omega/2\pi = \underline{4.7 \times 10^8 \text{ Hz}}$.

b) Specify the intrinsic impedance of the region $z > 0$ that would provide the appropriate reflected wave: Use

$$\Gamma = \frac{E_r}{E_{inc}} = \frac{50e^{j20^\circ}}{150} = \frac{1}{3}e^{j20^\circ} = 0.31 + j0.11 = \frac{\eta - \eta_0}{\eta + \eta_0}$$

13.5 (continued) Now

$$\eta = \eta_0 \left(\frac{1+\Gamma}{1-\Gamma} \right) = 377 \left(\frac{1+0.31+j0.11}{1-0.31-j0.31} \right) = \underline{691 + j177\ \Omega}$$

c) At what value of z ($-10\,\text{cm} < z < 0$) is the total electric field intensity a maximum amplitude? We found the phase of the reflection coefficient to be $\phi = 20° = .349$rad, and we use

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-.349}{20} = -0.017\,\text{m} = \underline{-1.7\,\text{cm}}$$

13.6. Region 1, $z < 0$, and region 2, $z > 0$, are described by the following parameters: $\epsilon_1' = 100$ pF/m, $\mu_1 = 25$ μH/m, $\epsilon_1'' = 0$, $\epsilon_2' = 200$ pF/m, $\mu_2 = 50$ μH/m, and $\epsilon_2''/\epsilon_2' = 0.5$. If $\mathbf{E}_1^+ = 5e^{-\alpha_1 z}\cos(4\times 10^9 t - \beta_1 z)\mathbf{a}_x$ V/m, find:

a) α_1: As $\epsilon_1'' = 0$, there is no loss mechanism that is modeled (see Eq. (44), Chapter 12), and so $\alpha_1 = \underline{0}$.

b) β_1: Since region 1 is lossless, the phase constant for the uniform plane wave will be

$$\beta_1 = \omega\sqrt{\mu_1\epsilon_1'} = (4\times 10^9)\sqrt{(25\times 10^{-6})(100\times 10^{-12})} = \underline{200\ \text{rad/m}}$$

c) $< \mathbf{S}_1^+ >$: To find the power density, we need the intrinsic impedance of region 1, given by

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1'}} = \sqrt{\frac{25\times 10^{-6}}{100\times 10^{-12}}} = 500\ \text{ohms}$$

Then the incident power density will be

$$< \mathbf{S}_1^+ >= \frac{1}{2\eta_1}|E_1|^2\,\mathbf{a}_z = \frac{5^2}{2(500)}\,\mathbf{a}_z = \underline{25\,\mathbf{a}_z\ \text{mW/m}^2}$$

d) $< \mathbf{S}_1^- >$: To find the reflected power, we need the intrinsic impedance of region 2. This is found using Eq. (48), Chapter 12:

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2'}}\ \frac{1}{\sqrt{1-j(\epsilon_2''/\epsilon_2')}} = \sqrt{\frac{50\times 10^{-6}}{200\times 10^{-12}}}\ \frac{1}{\sqrt{1-j0.5}} = 460 + j109\ \text{ohms}$$

Then the reflection coefficient at the 1-2 boundary is

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{460 + j109 - 500}{460 + j109 + 500} = -0.028 + j0.117$$

The reflected power fraction is then $|\Gamma|^2 = 1.44\times 10^{-2}$.
Therefore $< \mathbf{S}_1^- >= - < \mathbf{S}_1^+ > |\Gamma|^2 = \underline{-0.36\,\mathbf{a}_z\ \text{mW/m}^2}$.

13.6e) $< \mathbf{S}_2^+ >$: We first need the attenuation coefficient in region 2. This is given by Eq. (44) in Chapter 12, which in our case becomes

$$\alpha_2 = \omega\sqrt{\frac{\mu_2\epsilon_2'}{2}}\left[\sqrt{1+\left(\frac{\epsilon_2''}{\epsilon_2'}\right)^2}-1\right]^{1/2}$$

$$= (4\times 10^9)\left[\frac{(50\times 10^{-6})(200\times 10^{-12})}{2}\right]^{1/2}\left[\sqrt{1+0.25}-1\right]^{1/2} = 97.2 \text{ Np/m}$$

Now

$$< \mathbf{S}_2^+ > = < \mathbf{S}_1^+ > \left(1-|\Gamma|^2\right)e^{-2\alpha_2 z} = 25(0.986)e^{-2(97.2)z}\,\mathbf{a}_z = \underline{24.7e^{-194z}\,\mathbf{a}_z \text{ mW/m}^2}$$

Note the approximately 1 cm penetration depth.

13.7. The semi-infinite regions $z < 0$ and $z > 1$ m are free space. For $0 < z < 1$ m, $\epsilon_r' = 4$, $\mu_r = 1$, and $\epsilon_r'' = 0$. A uniform plane wave with $\omega = 4\times 10^8$ rad/s is travelling in the $\mathbf{a}_z$ direction toward the interface at $z = 0$.

a) Find the standing wave ratio in each of the three regions: First we find the phase constant in the middle region,

$$\beta_2 = \frac{\omega\sqrt{\epsilon_r'}}{c} = \frac{2(4\times 10^8)}{3\times 10^8} = 2.67\,\text{rad/m}$$

Then, with the middle layer thickness of 1 m, $\beta_2 d = 2.67\,\text{rad}$. Also, the intrinsic impedance of the middle layer is $\eta_2 = \eta_0/\sqrt{\epsilon_r'} = \eta_0/2$. We now find the input impedance:

$$\eta_{in} = \eta_2\left[\frac{\eta_0\cos(\beta_2 d) + j\eta_2\sin(\beta_2 d)}{\eta_2\cos(\beta_2 d) + j\eta_0\sin(\beta_2 d)}\right] = \frac{377}{2}\left[\frac{2\cos(2.67) + j\sin(2.67)}{\cos(2.67) + j2\sin(2.67)}\right] = 231 + j141$$

Now, at the first interface,

$$\Gamma_{12} = \frac{\eta_{in}-\eta_0}{\eta_{in}+\eta_0} = \frac{231 + j141 - 377}{231 + j141 + 377} = -.176 + j.273 = .325\angle 123^\circ$$

The standing wave ratio measured in region 1 is thus

$$s_1 = \frac{1+|\Gamma_{12}|}{1-|\Gamma_{12}|} = \frac{1+0.325}{1-0.325} = \underline{1.96}$$

In region 2 the standing wave ratio is found by considering the reflection coefficient for waves incident from region 2 on the second interface:

$$\Gamma_{23} = \frac{\eta_0 - \eta_0/2}{\eta_0 + \eta_0/2} = \frac{1-1/2}{1+1/2} = \frac{1}{3}$$

Then

$$s_2 = \frac{1+1/3}{1-1/3} = \underline{2}$$

Finally, $s_3 = \underline{1}$, since no reflected waves exist in region 3.

13.7b. Find the location of the maximum $|\mathbf{E}|$ for $z < 0$ that is nearest to $z = 0$. We note that the phase of Γ_{12} is $\phi = 123° = 2.15$ rad. Thus

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-2.15}{2(4/3)} = \underline{-.81\,\text{m}}$$

13.8. A wave starts at point a, propagates 100m through a lossy dielectric for which $\alpha = 0.5$ Np/m, reflects at normal incidence at a boundary at which $\Gamma = 0.3 + j0.4$, and then returns to point a. Calculate the ratio of the final power to the incident power after this round trip: Final power, P_f, and incident power, P_i, are related through

$$P_f = P_i e^{-2\alpha L}|\Gamma|^2 e^{-2\alpha L} \quad \Rightarrow \quad \frac{P_f}{P_i} = |0.3 + j0.4|^2 e^{-4(0.5)100} = \underline{3.5 \times 10^{-88}}(!)$$

Try measuring that.

13.9. Region 1, $z < 0$, and region 2, $z > 0$, are both perfect dielectrics ($\mu = \mu_0$, $\epsilon'' = 0$). A uniform plane wave traveling in the $\mathbf{a}_z$ direction has a radian frequency of 3×10^{10} rad/s. Its wavelengths in the two regions are $\lambda_1 = 5$ cm and $\lambda_2 = 3$ cm. What percentage of the energy incident on the boundary is

a) reflected; We first note that

$$\epsilon'_{r1} = \left(\frac{2\pi c}{\lambda_1 \omega}\right)^2 \quad \text{and} \quad \epsilon'_{r2} = \left(\frac{2\pi c}{\lambda_2 \omega}\right)^2$$

Therefore $\epsilon'_{r1}/\epsilon'_{r2} = (\lambda_2/\lambda_1)^2$. Then with $\mu = \mu_0$ in both regions, we find

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{1/\epsilon'_{r2}} - \eta_0\sqrt{1/\epsilon'_{r1}}}{\eta_0\sqrt{1/\epsilon'_{r2}} + \eta_0\sqrt{1/\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}/\epsilon'_{r2}} - 1}{\sqrt{\epsilon'_{r1}/\epsilon'_{r2}} + 1} = \frac{(\lambda_2/\lambda_1) - 1}{(\lambda_2/\lambda_1) + 1}$$

$$= \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{3-5}{3+5} = -\frac{1}{4}$$

The fraction of the incident energy that is reflected is then $|\Gamma|^2 = 1/16 = \underline{6.25 \times 10^{-2}}$.

b) transmitted? We use part a and find the transmitted fraction to be $1 - |\Gamma|^2 = 15/16 = \underline{0.938}$.

c) What is the standing wave ratio in region 1? Use

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1/4}{1 - 1/4} = \frac{5}{3} = \underline{1.67}$$

13.10. In Fig. 13.1, let region 2 be free space, while $\mu_{r1} = 1$, $\epsilon''_{r1} = 0$, and ϵ'_{r1} is unknown. Find ϵ'_{r1} if

a) the amplitude of $\mathbf{E}_1^-$ is one-half that of $\mathbf{E}_1^+$: Since region 2 is free space, the reflection coefficient is

$$\Gamma = \frac{|\mathbf{E}_1^-|}{|\mathbf{E}_1^+|} = \frac{\eta_0 - \eta_1}{\eta_0 + \eta_1} = \frac{\eta_0 - \eta_0/\sqrt{\epsilon'_{r1}}}{\eta_0 + \eta_0/\sqrt{\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1} = \frac{1}{2} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{9}$$

.

b) $< \mathbf{S}_1^- >$ is one-half of $< \mathbf{S}_1^+ >$: This time

$$|\Gamma|^2 = \left| \frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1} \right|^2 = \frac{1}{2} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{34}$$

c) $|\mathbf{E}_1|_{min}$ is one-half $|\mathbf{E}_1|_{max}$: Use

$$\frac{|\mathbf{E}_1|_{max}}{|\mathbf{E}_1|_{min}} = s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2 \quad \Rightarrow \quad |\Gamma| = \Gamma = \frac{1}{3} = \frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{4}$$

13.11. A 150 MHz uniform plane wave in normally-incident from air onto a material whose intrinsic impedance is unknown. Measurements yield a standing wave ratio of 3 and the appearance of an electric field minimum at 0.3 wavelengths in front of the interface. Determine the impedance of the unknown material: First, the field minimum is used to find the phase of the reflection coefficient, where

$$z_{min} = -\frac{1}{2\beta}(\phi + \pi) = -0.3\lambda \quad \Rightarrow \quad \phi = 0.2\pi$$

where $\beta = 2\pi/\lambda$ has been used. Next,

$$|\Gamma| = \frac{s - 1}{s + 1} = \frac{3 - 1}{3 + 1} = \frac{1}{2}$$

So we now have

$$\Gamma = 0.5e^{j0.2\pi} = \frac{\eta_u - \eta_0}{\eta_u + \eta_0}$$

We solve for η_u to find

$$\eta_u = \eta_0(1.70 + j1.33) = \underline{641 + j501\ \Omega}$$

13.12. A 50MHz uniform plane wave is normally incident from air onto the surface of a calm ocean. For seawater, $\sigma = 4$ S/m, and $\epsilon'_r = 78$.

a) Determine the fractions of the incident power that are reflected and transmitted: First we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50 \times 10^6)(78)(8.854 \times 10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 65, Chapter 11), the intrinsic impedance is $\eta_s = \sqrt{\pi f \mu/\sigma}(1 + j)$, and the reflection coefficient becomes

$$\Gamma = \frac{\sqrt{\pi f \mu/\sigma}\ (1 + j) - \eta_0}{\sqrt{\pi f \mu/\sigma}\ (1 + j) + \eta_0}$$

13.12 (continued) where $\sqrt{\pi f \mu / \sigma} = \sqrt{\pi(50 \times 10^6)(4\pi \times 10^{-7})/4} = 7.0$. The fraction of the power reflected is

$$\frac{P_r}{P_i} = |\Gamma|^2 = \frac{[\sqrt{\pi f \mu/\sigma} - \eta_0]^2 + \pi f \mu/\sigma}{[\sqrt{\pi f \mu/\sigma} + \eta_0]^2 + \pi f \mu/\sigma} = \frac{[7.0 - 377]^2 + 49.0}{[7.0 + 377]^2 + 49.0} = \underline{0.93}$$

The transmitted fraction is then

$$\frac{P_t}{P_i} = 1 - |\Gamma|^2 = 1 - 0.93 = \underline{0.07}$$

b) Qualitatively, how will these answers change (if at all) as the frequency is increased? Within the limits of our good conductor approximation (loss tangent greater than about ten), the reflected power fraction, using the formula derived in part a, is found to decrease with increasing frequency. The transmitted power fraction thus increases.

13.13. A right-circularly-polarized plane wave is normally incident from air onto a semi-infinite slab of plexiglas ($\epsilon_r' = 3.45$, $\epsilon_r'' = 0$). Calculate the fractions of the incident power that are reflected and transmitted. Also, describe the polarizations of the reflected and transmitted waves. First, the impedance of the plexiglas will be $\eta = \eta_0/\sqrt{3.45} = 203\,\Omega$. Then

$$\Gamma = \frac{203 - 377}{203 + 377} = -0.30$$

The reflected power fraction is thus $|\Gamma|^2 = \underline{0.09}$. The total electric field in the plane of the interface must rotate in the same direction as the incident field, in order to continually satisfy the boundary condition of tangential electric field continuity across the interface. Therefore, the reflected wave will have to be left circularly polarized in order to make this happen. The transmitted power fraction is now $1 - |\Gamma|^2 = \underline{0.91}$. The transmitted field will be right circularly polarized (as the incident field) for the same reasons.

13.14. A left-circularly-polarized plane wave is normally-incident onto the surface of a perfect conductor.

a) Construct the superposition of the incident and reflected waves in phasor form: Assume positive z travel for the incident electric field. Then, with reflection coefficient, $\Gamma = -1$, the incident and reflected fields will add to give the total field:

$$\mathbf{E}_{tot} = \mathbf{E}_i + \mathbf{E}_r = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} - E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{+j\beta z}$$

$$= E_0 \left[\underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)} \mathbf{a}_x + j \underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)} \mathbf{a}_y \right] = \underline{2E_0 \sin(\beta z)\,[\mathbf{a}_y - j\mathbf{a}_x]}$$

b) Determine the real instantaneous form of the result of part a:

$$\mathbf{E}(z,t) = \text{Re}\left\{\mathbf{E}_{tot} e^{j\omega t}\right\} = \underline{2E_0 \sin(\beta z)\,[\cos(\omega t)\mathbf{a}_y + \sin(\omega t)\mathbf{a}_x]}$$

c) Describe the wave that is formed: This is a standing wave exhibiting circular polarization in time. At each location along the z axis, the field vector rotates clockwise in the xy plane, and has amplitude (constant with time) given by $2E_0 \sin(\beta z)$.

13.15. Consider these regions in which $\epsilon'' = 0$: region 1, $z < 0$, $\mu_1 = 4\,\mu$H/m and $\epsilon'_1 = 10$ pF/m; region 2, $0 < z < 6$ cm, $\mu_2 = 2\,\mu H/m$, $\epsilon'_2 = 25$ pF/m; region 3, $z > 6$ cm, $\mu_3 = \mu_1$ and $\epsilon'_3 = \epsilon'_1$.

a) What is the lowest frequency at which a uniform plane wave incident from region 1 onto the boundary at $z = 0$ will have no reflection? This frequency gives the condition $\beta_2 d = \pi$, where $d = 6$ cm, and $\beta_2 = \omega\sqrt{\mu_2 \epsilon'_2}$ Therefore

$$\beta_2 d = \pi \quad \Rightarrow \quad \omega = \frac{\pi}{(.06)\sqrt{\mu_2 \epsilon'_2}} \quad \Rightarrow \quad f = \frac{1}{0.12\sqrt{(2\times10^{-6})(25\times10^{-12})}} = \underline{1.2\text{ GHz}}$$

b) If $f = 50$ MHz, what will the standing wave ratio be in region 1? At the given frequency, $\beta_2 = (2\pi\times5\times10^7)\sqrt{(2\times10^{-6})(25\times10^{-12})} = 2.22$ rad/m. Thus $\beta_2 d = 2.22(.06) = 0.133$. The intrinsic impedance of regions 1 and 3 is $\eta_1 = \eta_3 = \sqrt{(4\times10^{-6})/(10^{-11})} = 632\,\Omega$. The input impedance at the first interface is now

$$\eta_{in} = 283\left[\frac{632\cos(.133) + j283\sin(.133)}{283\cos(.133) + j632\sin(.133)}\right] = 589 - j138 = 605\angle - .23$$

The reflection coefficient is now

$$\Gamma = \frac{\eta_{in} - \eta_1}{\eta_{in} + \eta_1} = \frac{589 - j138 - 632}{589 - j138 + 632} = .12\angle - 1.7$$

The standing wave ratio is now

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + .12}{1 - .12} = \underline{1.27}$$

13.16. A uniform plane wave in air is normally-incident onto a lossless dielectric plate of thickness $\lambda/8$, and of intrinsic impedance $\eta = 260\ \Omega$. Determine the standing wave ratio in front of the plate. Also find the fraction of the incident power that is transmitted to the other side of the plate: With the a thickness of $\lambda/8$, we have $\beta d = \pi/4$, and so $\cos(\beta d) = \sin(\beta d) = 1\sqrt{2}$. The input impedance thus becomes

$$\eta_{in} = 260\left[\frac{377 + j260}{260 + j377}\right] = 243 - j92\ \Omega$$

The reflection coefficient is then

$$\Gamma = \frac{(243 - j92) - 377}{(243 - j92) + 377} = -0.19 - j0.18 = 0.26\angle - 2.4\text{rad}$$

Therefore

$$s = \frac{1 + .26}{1 - .26} = \underline{1.7} \quad \text{and} \quad 1 - |\Gamma|^2 = 1 - (.26)^2 = \underline{0.93}$$

13.17. Repeat Problem 13.16 for the cases in which the frequency is

a) doubled: If this is true, then $d = \lambda/4$, and thus $\eta_{in} = (260)^2/377 = 179$. The reflection coefficient becomes

$$\Gamma = \frac{179 - 377}{179 + 377} = -0.36 \quad \Rightarrow \quad s = \frac{1 + .36}{1 - .36} = \underline{2.13}$$

Then $1 - |\Gamma|^2 = 1 - (.36)^2 = \underline{0.87}$.

b) quadrupled: Now, $d = \lambda/2$, and so we have a half-wave section surrounded by air. Transmission will be total, and so $s = \underline{1}$ and $1 - |\Gamma|^2 = \underline{1}$.

13.18. A uniform plane wave is normally-incident onto a slab of glass ($n = 1.45$) whose back surface is in contact with a perfect conductor. Determine the reflective phase shift at the front surface of the glass if the glass thickness is: (a) $\lambda/2$; (b) $\lambda/4$; (c) $\lambda/8$.

With region 3 being a perfect conductor, $\eta_3 = 0$, and Eq. (36) gives the input impedance to the structure as $\eta_{in} = j\eta_2 \tan\beta\ell$. The reflection coefficient is then

$$\Gamma = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{j\eta_2 \tan\beta\ell - \eta_0}{j\eta_2 \tan\beta\ell + \eta_0} = \frac{\eta_2^2 \tan^2\beta\ell - \eta_0^2 + j2\eta_0\eta_2 \tan\beta\ell}{\eta_2^2 \tan^2\beta\ell + \eta_0^2} = \Gamma_r + j\Gamma_i$$

where the last equality occurs by multiplying the numerator and denominator of the middle term by the complex conjugate of its denominator. The reflective phase is now

$$\phi = \tan^{-1}\left(\frac{\Gamma_i}{\Gamma_r}\right) = \tan^{-1}\left[\frac{2\eta_2\eta_0 \tan\beta\ell}{\eta_2^2 \tan^2\beta\ell - \eta_0^2}\right] = \tan^{-1}\left[\frac{(2.90)\tan\beta\ell}{\tan\beta\ell - 2.10}\right]$$

where $\eta_2 = \eta_0/1.45$ has been used. We can now evaluate the phase shift for the three given cases. First, when $\ell = \lambda/2$, $\beta\ell = \pi$, and thus $\underline{\phi(\lambda/2) = 0}$. Next, when $\ell = \lambda/4$, $\beta\ell = \pi/2$, and

$$\phi(\lambda/4) \to \tan^{-1}[2.90] = \underline{71^\circ}$$

as $\ell \to \lambda/4$. Finally, when $\ell = \lambda/8$, $\beta\ell = \pi/4$, and

$$\phi(\lambda/8) = \tan^{-1}\left[\frac{2.90}{1 - 2.10}\right] = \underline{-69.2^\circ} \text{ (or } 291^\circ\text{)}$$

13.19. You are given four slabs of lossless dielectric, all with the same intrinsic impedance, η, known to be different from that of free space. The thickness of each slab is $\lambda/4$, where λ is the wavelength as measured in the slab material. The slabs are to be positioned parallel to one another, and the combination lies in the path of a uniform plane wave, normally-incident. The slabs are to be arranged such that the air spaces between them are either zero, one-quarter wavelength, or one-half wavelength in thickness. Specify an arrangement of slabs and air spaces such that

a) the wave is totally transmitted through the stack: In this case, we look for a combination of half-wave sections. Let the inter-slab distances be d_1, d_2, and d_3 (from left to right). Two possibilities are i.) $\underline{d_1 = d_2 = d_3 = 0}$, thus creating a single section of thickness λ, or ii.) $\underline{d_1 = d_3 = 0,\ d_2 = \lambda/2}$, thus yielding two half-wave sections separated by a half-wavelength.

b) the stack presents the highest reflectivity to the incident wave: The best choice here is to make $\underline{d_1 = d_2 = d_3 = \lambda/4}$. Thus every thickness is one-quarter wavelength. The impedances transform as follows: First, the input impedance at the front surface of the last slab (slab 4) is $\eta_{in,1} = \eta^2/\eta_0$. We transform this back to the back surface of slab 3, moving through a distance of $\lambda/4$ in free space: $\eta_{in,2} = \eta_0^2/\eta_{in,1} = \eta_0^3/\eta^2$. We next transform this impedance to the front surface of slab 3, producing $\eta_{in,3} = \eta^2/\eta_{in,2} = \eta^4/\eta_0^3$. We continue in this manner until reaching the front surface of slab 1, where we find $\eta_{in,7} = \eta^8/\eta_0^7$. Assuming $\eta < \eta_0$, the ratio η^n/η_0^{n-1} becomes smaller as n increases (as the number of slabs increases). The reflection coefficient for waves incident on the front slab thus gets close to unity, and approaches 1 as the number of slabs approaches infinity.

13.20. The 50MHz plane wave of Problem 13.12 is incident onto the ocean surface at an angle to the normal of 60°. Determine the fractions of the incident power that are reflected and transmitted for

a) s polarization: To review Problem 12, we first we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50 \times 10^6)(78)(8.854 \times 10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 65, Chapter 11), and with $\mu = \mu_0$, the intrinsic impedance is $\eta_s = \sqrt{\pi f \mu/\sigma}(1 + j) = 7.0(1 + j)$. Next we need the angle of refraction, which means that we need to know the refractive index of seawater at 50MHz. For a uniform plane wave in a good conductor, the phase constant is

$$\beta = \frac{n_{sea}\,\omega}{c} \doteq \sqrt{\pi f \mu \sigma} \quad \Rightarrow \quad n_{sea} \doteq c\sqrt{\frac{\mu\sigma}{4\pi f}} = 26.8$$

Then, using Snell's law, the angle of refraction is found:

$$\sin\theta_2 = \frac{n_{sea}}{n_1}\sin\theta_1 = 26.8\sin(60^\circ) \quad \Rightarrow \quad \theta_2 = 1.9^\circ$$

This angle is small enough so that $\cos\theta_2 \doteq 1$. Therefore, for s polarization,

$$\Gamma_s \doteq \frac{\eta_{s2} - \eta_{s1}}{\eta_{s2} + \eta_{s1}} = \frac{7.0(1 + j) - 377/\cos 60^\circ}{7.0(1 + j) + 377/\cos 60^\circ} = -0.98 + j0.018 = 0.98\angle 179^\circ$$

3.20a (continued) The fraction of the power reflected is now $|\Gamma_s|^2 = \underline{0.96}$. The fraction transmitted is then 0.04.

b) p polarization: Again, with the refracted angle close to zero, the relection coefficient for p polarization is

$$\Gamma_p \doteq \frac{\eta_{p2} - \eta_{p1}}{\eta_{p2} + \eta_{p1}} = \frac{7.0(1+j) - 377\cos 60^\circ}{7.0(1+j) + 377\cos 60^\circ} = -0.93 + j0.069 = 0.93\angle 176^\circ$$

The fraction of the power reflected is now $|\Gamma_p|^2 = \underline{0.86}$. The fraction transmitted is then 0.14.

13.21. A right-circularly polarized plane wave in air is incident at Brewster's angle onto a semi-infinite slab of plexiglas ($\epsilon'_r = 3.45$, $\epsilon''_r = 0$, $\mu = \mu_0$).

a) Determine the fractions of the incident power that are reflected and transmitted: In plexiglas, Brewster's angle is $\theta_B = \theta_1 = \tan^{-1}(\epsilon'_{r2}/\epsilon'_{r1}) = \tan^{-1}(\sqrt{3.45}) = 61.7^\circ$. Then the angle of refraction is $\theta_2 = 90^\circ - \theta_B$ (see Example 13.9), or $\theta_2 = 28.3^\circ$. With incidence at Brewster's angle, all *p*-polarized power will be transmitted — only *s*-polarized power will be reflected. This is found through

$$\Gamma_s = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}} = \frac{.614\eta_0 - 2.11\eta_0}{.614\eta_0 + 2.11\eta_0} = -0.549$$

where $\eta_{1s} = \eta_1 \sec\theta_1 = \eta_0 \sec(61.7^\circ) = 2.11\eta_0$,
and $\eta_{2s} = \eta_2 \sec\theta_2 = (\eta_0/\sqrt{3.45})\sec(28.3^\circ) = 0.614\eta_0$. Now, the reflected power fraction is $|\Gamma|^2 = (-.549)^2 = .302$. Since the wave is circularly-polarized, the *s*-polarized component represents one-half the total incident wave power, and so the fraction of the *total* power that is reflected is $.302/2 = 0.15$, or 15%. The fraction of the incident power that is transmitted is then the remainder, or 85%.

b) Describe the polarizations of the reflected and transmitted waves: Since all the *p*-polarized component is transmitted, the reflected wave will be entirely *s*-polarized (linear). The transmitted wave, while having all the incident *p*-polarized power, will have a reduced *s*-component, and so this wave will be right-elliptically polarized.

13.22. A dielectric waveguide is shown in Fig. 13.16 with refractive indices as labeled. Incident light enters the guide at angle ϕ from the front surface normal as shown. Once inside, the light totally reflects at the upper $n_1 - n_2$ interface, where $n_1 > n_2$. All subsequent reflections from the upper an lower boundaries will be total as well, and so the light is confined to the guide. Express, in terms of n_1 and n_2, the maximum value of ϕ such that total confinement will occur, with $n_0 = 1$. The quantity $\sin\phi$ is known as the *numerical aperture* of the guide.

From the illustration we see that ϕ_1 maximizes when θ_1 is at its minimum value. This minimum will be the critical angle for the $n_1 - n_2$ interface, where $\sin\theta_c = \sin\theta_1 = n_2/n_1$. Let the refracted angle to the right of the vertical interface (not shown) be ϕ_2, where $n_0\sin\phi_1 = n_1\sin\phi_2$. Then we see that $\phi_2 + \theta_1 = 90^\circ$, and so $\sin\theta_1 = \cos\phi_2$. Now, the numerical aperture becomes

$$\sin\phi_{1max} = \frac{n_1}{n_0}\sin\phi_2 = n_1\cos\theta_1 = n_1\sqrt{1-\sin^2\theta_1} = n_1\sqrt{1-(n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}$$

Finally, $\phi_{1max} = \underline{\sin^{-1}\left(\sqrt{n_1^2 - n_2^2}\right)}$ is the numerical aperture angle.

13.23. Suppose that ϕ_1 in Fig. 13.16 is Brewster's angle, and that θ_1 is the critical angle. Find n_0 in terms of n_1 and n_2: With the incoming ray at Brewster's angle, the refracted angle of this ray (measured from the inside normal to the front surface) will be $90° - \phi_1$. Therefore, $\phi_1 = \theta_1$, and thus $\sin\phi_1 = \sin\theta_1$. Thus

$$\sin\phi_1 = \frac{n_1}{\sqrt{n_0^2 + n_1^2}} = \sin\theta_1 = \frac{n_2}{n_1} \quad \Rightarrow \quad n_0 = \underline{(n_1/n_2)\sqrt{n_1^2 - n_2^2}}$$

Alternatively, we could have used the result of Problem 13.22, in which it was found that $\sin\phi_1 = (1/n_0)\sqrt{n_1^2 - n_2^2}$, which we then set equal to $\sin\theta_1 = n_2/n_1$ to get the same result.

13.24. A *Brewster prism* is designed to pass p-polarized light without any reflective loss. The prism of Fig. 13.17 is made of glass ($n = 1.45$), and is in air. Considering the light path shown, determine the apex angle, α: With entrance and exit rays at Brewster's angle (to eliminate reflective loss), the interior ray must be horizontal, or parallel to the bottom surface of the prism. From the geometry, the angle between the interior ray and the normal to the prism surfaces that it intersects is $\alpha/2$. Since this angle is also Brewster's angle, we may write:

$$\alpha = 2\sin^{-1}\left(\frac{1}{\sqrt{1+n^2}}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{1+(1.45)^2}}\right) = 1.21\,\text{rad} = \underline{69.2°}$$

13.25. In the Brewster prism of Fig. 13.17, determine for s-polarized light the fraction of the incident power that is transmitted through the prism: We use $\Gamma_s = (\eta_{s2} - \eta_{s1})/(\eta_{s2} + \eta_{s1})$, where

$$\eta_{s2} = \frac{\eta_2}{\cos(\theta_{B2})} = \frac{\eta_2}{n/\sqrt{1+n^2}} = \frac{\eta_0}{n^2}\sqrt{1+n^2}$$

and

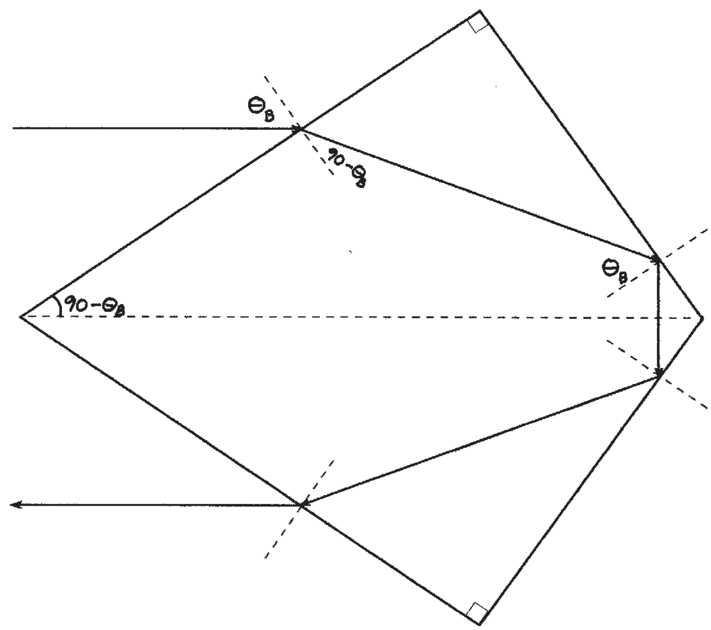
$$\eta_{s1} = \frac{\eta_1}{\cos(\theta_{B1})} = \frac{\eta_1}{1/\sqrt{1+n^2}} = \eta_0\sqrt{1+n^2}$$

Thus, at the first interface, $\Gamma = (1-n^2)/(1+n^2)$. At the second interface, Γ will be equal but of opposite sign to the above value. The power transmission coefficient through each interface is $1 - |\Gamma|^2$, so that for both interfaces, we have, with $n = 1.45$:

$$\frac{P_{tr}}{P_{inc}} = \left(1 - |\Gamma|^2\right)^2 = \left[1 - \left(\frac{n^2-1}{n^2+1}\right)^2\right]^2 = \underline{0.76}$$

13.26. Show how a single block of glass can be used to turn a p-polarized beam of iight through 180°, with the light suffering, in principle, zero reflective loss. The light is incident from air, and the returning beam (also in air) may be displaced sideways from the incident beam. Specify all pertinent angles and use $n = 1.45$ for glass. More than one design is possible here.

The prism below is designed such that light enters at Brewster's angle, and once inside, is turned around using total reflection. Using the result of Example 13.9, we find that with glass, $\theta_B = 55.4°$, which, by the geometry, is also the incident angle for total reflection at the back of the prism. For this to work, the Brewster angle must be greater than or equal to the critical angle. This is in fact the case, since $\theta_c = \sin^{-1}(n_2/n_1) = \sin^{-1}(1/1.45) = 43.6°$.



13.27. Using Eq. (79) in Chapter 12 as a starting point, determine the ratio of the group and phase velocities of an electromagnetic wave in a good conductor. Assume conductivity does not vary with frequency: In a good conductor:

$$\beta = \sqrt{\pi f \mu \sigma} = \sqrt{\frac{\omega\mu\sigma}{2}} \qquad \rightarrow \qquad \frac{d\beta}{d\omega} = \frac{1}{2}\left[\frac{\omega\mu\sigma}{2}\right]^{-1/2} \frac{\mu\sigma}{2}$$

Thus

$$\frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^{-1} = 2\sqrt{\frac{2\omega}{\mu\sigma}} = v_g \qquad \text{and} \qquad v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega\mu\sigma/2}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Therefore $v_g/v_p = \underline{2}$.

13.28. Over a small wavelength range, the refractive index of a certain material varies approximately linearly with wavelength as $n(\lambda) \doteq n_a + n_b(\lambda - \lambda_a)$, where n_a, n_b, and λ_a are constants, and where λ is the free space wavelength.

a) Show that $d/d\omega = -(2\pi c/\omega^2)d/d\lambda$: With λ as the free space wavelength, we use $\lambda = 2\pi c/\omega$, from which $d\lambda/d\omega = -2\pi c/\omega^2$. Then $d/d\omega = (d\lambda/d\omega)\, d/d\lambda = -(2\pi c/\omega^2)\, d/d\lambda$.

b) Using $\beta(\lambda) = 2\pi n/\lambda$, determine the wavelength-dependent (or independent) group delay over a unit distance: This will be

$$
\begin{aligned}
t_g = \frac{1}{v_g} = \frac{d\beta}{d\omega} &= \frac{d}{d\omega}\left[\frac{2\pi n(\lambda)}{\lambda}\right] = -\frac{2\pi c}{\omega^2}\frac{d}{d\lambda}\left[\frac{2\pi}{\lambda}\left[n_a + n_b(\lambda - \lambda_a)\right]\right] \\
&= -\frac{2\pi c}{\omega^2}\left[-\frac{2\pi}{\lambda^2}\left[n_a + n_b(\lambda - \lambda_a)\right] + \frac{2\pi}{\lambda}n_b\right] \\
&= -\frac{\lambda^2}{2\pi c}\left[-\frac{2\pi n_a}{\lambda^2} + \frac{2\pi n_b \lambda_a}{\lambda^2}\right] = \underline{\frac{1}{c}(n_a - n_b\lambda_a)\ \text{s/m}}
\end{aligned}
$$

c) Determine β_2 from your result of part *b*: $\beta_2 = d^2\beta/d\omega^2|_{\omega_0}$. Since the part *b* result is independent of wavelength (and of frequency), it follows that $\underline{\beta_2 = 0}$.

d) Discuss the implications of these results, if any, on pulse broadening: A wavelength-independent group delay (leading to zero β_2) means that there will simply be no pulse broadening at all. All frequency components arrive simultaneously. This sort of thing happens in most transparent materials – that is, there will be a certain wavelength, known as the *zero dispersion wavelength*, around which the variation of n with λ is locally linear. Transmitting pulses at this wavelength will result in no pulse broadening (to first order).

13.29. A $T = 5$ ps transform-limited pulse propagates in a dispersive channel for which $\beta_2 = 10\ \text{ps}^2/\text{km}$. Over what distance will the pulse spread to twice its initial width? After propagation, the width is $T' = \sqrt{T^2 + (\Delta\tau)^2} = 2T$. Thus $\Delta\tau = \sqrt{3}T$, where $\Delta\tau = \beta_2 z/T$. Therefore

$$
\frac{\beta_2 z}{T} = \sqrt{3}T \text{ or } z = \frac{\sqrt{3}T^2}{\beta_2} = \frac{\sqrt{3}(5\,\text{ps})^2}{10\,\text{ps}^2/\text{km}} = \underline{4.3\ \text{km}}
$$

13.30. A $T = 20$ ps transform-limited pulse propagates through 10 km of a dispersive channel for which $\beta_2 = 12\,\text{ps}^2/\text{km}$. The pulse then propagates through a second 10 km channel for which $\beta_2 = -12\,\text{ps}^2/\text{km}$. Describe the pulse at the output of the second channel and give a physical explanation for what happened.

Our theory of pulse spreading will allow for changes in β_2 down the length of the channel. In fact, we may write in general:

$$
\Delta\tau = \frac{1}{T}\int_0^L \beta_2(z)\, dz
$$

Having β_2 change sign at the midpoint, yields a zero $\Delta\tau$, and so the pulse emerges from the output unchanged! Physically, the pulse acquires a positive linear chirp (frequency increases with time over the pulse envelope) during the first half of the channel. When β_2 switches sign, the pulse begins to acquire a negative chirp in the second half, which, over an equal distance, will completely eliminate the chirp acquired during the first half. The pulse, if originally transform-limited at input, will emerge, again transform-limited, at its original width. More generally, complete *dispersion compensation* is achieved using a two-segment channel when $\beta_2 L = -\beta_2' L'$, assuming dispersion terms of higher order than β_2 do not exist.