4. **(Bonus question:** 5 marks) Consider again the signal $x(t)$ in the previous question. What proportion of the total signal power is contained in the frequency range $|\omega| \leq 3\pi$? Recall that Parseval's theorem states that

$$\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(\pi)t} = \cdots \; c_{-4} e^{-j4\pi t} + c_3 e^{-j3\pi t} + c_2 e^{-j2\pi t} + c_1 e^{-j\pi t} + c_0 + c_4 e^{j4\pi t} + c_3 e^{j3\pi t} + c_2 e^{j2\pi t} + c_1 e^{j\pi t}$$

$$T = 2 \quad , \quad c_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right) & o.w \end{cases}$$

$$\frac{1}{2}\int_{-1}^{1} x^2(t)dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2}\int_{-1/2}^{1/2} 1\, dt = \frac{1}{2}$$ انرژی کل سیگنال

$$\sum_{k=-3}^{3} |c_k|^2 = \left|\frac{1}{-3\pi}\sin\left(\frac{-3\pi}{2}\right)\right|^2 + \left|\frac{1}{-2\pi}\sin\left(\frac{-2\pi}{2}\right)\right|^2 + \left|\frac{1}{-\pi}\sin\left(\frac{-\pi}{2}\right)\right|^2 + \left(\frac{1}{2}\right)^2$$

$$\left|\frac{1}{3\pi}\sin\left(\frac{-3\pi}{2}\right)\right|^2 + \left|\frac{1}{2\pi}\sin\left(\frac{-2\pi}{2}\right)\right|^2 + \left|\frac{1}{\pi}\sin\left(\frac{\pi}{2}\right)\right|^2$$

$$= \left(\frac{1}{4}\right) + 2\left(\frac{1}{9\pi^2}\sin^2\left(\frac{3\pi}{2}\right)\right) + 2\left(\frac{1}{4\pi^2}\sin^2(\pi)\right) + 2\left(\frac{1}{\pi^2}\sin^2\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{4} + 2\left(\frac{1}{9\pi^2} + \frac{1}{\pi^2}\right) = \frac{1}{4} + 2\left(\frac{10}{9\pi^2}\right) = 0.475$$

$\frac{1}{2}$ 100

0.475 $x = 95.03\%$

جواب

٪95 انرژی کل سیگنال در همان ۳ هارمونی اول قرار گرفته است به همین دلیل است که در اغلب موارد ۳ هارمونی ها
بالا صرفنظر می کنند

ـ نشان دهید که هر سه سیستم LTI داده شده با پاسخ ضربه های $h_1(t)$ ، $h_2(t)$ ، $h_3(t)$ هر سه به ورودی $x(t)=\cos(t)$ پاسخ یکسانی دارند.

$$h_1(t)=u(t), \quad h_2(t)=-2\delta(t)+5e^{-2t}u(t)$$

$$h_3(t)=2te^{-t}u(t)$$

ب) پیدا کنید پاسخ ضربه سیستم LTI دیگری را که باز هم به ورودی $x(t)=\cos(t)$ پاسخ یکسان دهد.

$$h_1(t)=u(t) \Rightarrow H_1(j\omega)=\frac{1}{j\omega}+\pi\delta(\omega)$$

$$x(t)=\cos(t)=\frac{1}{2}e^{jt}+\frac{1}{2}e^{-jt} \Rightarrow y_1(t)=\frac{1}{2}\left(\frac{1}{j1}\right)e^{jt}+\frac{1}{2}\left(\frac{1}{-j1}\right)e^{-jt}$$

$$y_1(t)=\sin(t)$$

$$h_2(t)=-2\delta(t)+5e^{-2t}u(t) \Rightarrow H_2(j\omega)=-2+\frac{5}{2+j\omega}$$

$$x(t)=\frac{1}{2}e^{jt}+\frac{1}{2}e^{-jt} \Rightarrow y_2(t)=\frac{1}{2}\left(-2+\frac{5}{2+j1}\right)e^{jt}+\frac{1}{2}\left(-2+\frac{5}{2-j1}\right)e^{-jt}$$

$$=\frac{1}{2}\left(-2+\frac{5(2-j1)}{5}\right)e^{jt}+\frac{1}{2}\left(-2+\frac{5(2+j1)}{5}\right)e^{-jt}$$

$$y_2(t)=\frac{-j1}{2}e^{jt}+\frac{j1}{2}e^{-jt}=\sin(t)$$

$$h_3(t)=2te^{-t}u(t) \Rightarrow H_3(j\omega)=\frac{2}{(1+j\omega)^2}$$

$$x(t)=\frac{1}{2}e^{jt}+\frac{1}{2}e^{-jt} \Rightarrow y_3(t)=\frac{1}{2}\left(\frac{2}{1+j}\right)^2 e^{jt}+\frac{1}{2}\left(\frac{2}{1-j}\right)^2 e^{-jt}$$

$$=\frac{1}{j2}e^{jt}+\frac{1}{-j2}e^{-jt}=\sin(t)$$

$$y_1(t)=h_1(t)*x(t)=\sin(t) \qquad y_2(t)=h_2(t)*x(t)=\sin(t), \quad y_3(t)=h_3(t)*x(t)=\sin(t)$$

$$\Rightarrow y_1(t)+y_2(t)=h_1(t)*x(t)+h_2(t)*x(t)=2\sin(t)$$

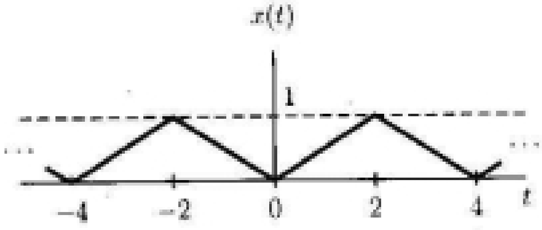
$$\Rightarrow \left(\frac{1}{2}h_1(t)+\frac{1}{2}h_2(t)\right)*x(t)=\sin(t) \Rightarrow h_4(t)=\frac{1}{2}\left(h_1(t)+h_2(t)\right)$$

$$h_5(t)=\frac{1}{2}\left(h_1(t)+h_3(t)\right)$$

$$\Rightarrow \qquad h_6(t)=\frac{1}{2}\left(h_2(t)+h_3(t)\right)$$

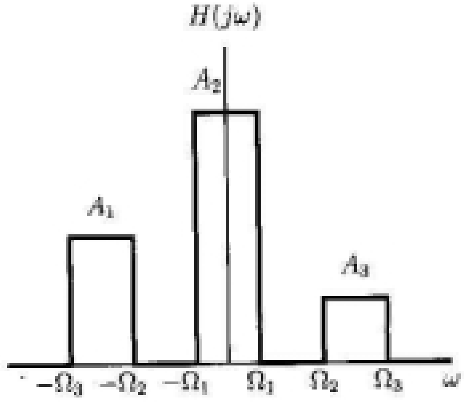
$$h_7(t)=\frac{1}{3}\left(h_1(t)+h_2(t)+h_3(t)\right)$$

Problem 2 The periodic triangular wave shown below has Fourier series coefficients a_k.



$$a_k = \begin{cases} 2\dfrac{\sin(k\pi/2)}{j(k\pi)^2}e^{-jk\pi/2}, & k \neq 0 \\ \dfrac{1}{2}, & k = 0. \end{cases}$$

Consider the LTI system with frequency response $H(j\omega)$ depicted below:



Determine values of A_1, A_2, A_3, Ω_1, Ω_2, and Ω_3 of the LTI filter $H(j\omega)$ such that

$$y(t) = 1 - \cos\left(\frac{3\pi}{2}t\right).$$

$y(t) = 1 - \frac{1}{2}e^{j\frac{3\pi}{2}t} - \frac{1}{2}e^{-j\frac{3\pi}{2}t}$ $\quad T = 4$

$y(t) = 1 - \frac{1}{2}e^{j3(\frac{2\pi}{4})t} - \frac{1}{2}e^{-j3(\frac{2\pi}{4})t}$ $\quad b_0 = 1,\ b_3 = \frac{-1}{2}$

$b_{-3} = \frac{-1}{2}$

$b_0 = a_0 \times H(j0) = \frac{1}{2} \times A_2 = 1 \Rightarrow A_2 = 2$

$b_3 = a_3 \cdot H(j3\frac{2\pi}{4}) = a_3 \cdot H(j\frac{3\pi}{2}) = 2\frac{\sin(\frac{3\pi}{2})}{j(3\pi)^2} \cdot e^{-j\frac{3\pi}{2}} \cdot H(j\frac{3\pi}{2}) = \frac{-1}{2}$

$= \frac{-2}{j(3\pi)^2} \cdot -j \times -1 \times H(j\frac{3\pi}{2}) = \frac{-1}{2}$

$H(j\frac{3\pi}{2}) = \frac{(3\pi)^2}{4} = A_3,\ \Omega_2 < \frac{3\pi}{2} < \Omega_3$

$b_{-3} = 2\frac{\sin(-\frac{3\pi}{2})}{j(-3\pi)^2} \cdot e^{j\frac{3\pi}{2}} \cdot H(-j\frac{3\pi}{2}) = \frac{2}{j(3\pi)^2} \cdot -j\, H(-j\frac{3\pi}{2}) = \frac{-1}{2} \Rightarrow H(-j\frac{3\pi}{2}) = \frac{(3\pi)^2}{4} = A_1 = A_3$

$b_{\pm 1} = 0 \Rightarrow H(\pm j\frac{2\pi}{4}) = 0 \quad \Omega_1 < \frac{2\pi}{4}$

$b_{\pm 2} = 0 \quad H(\pm j\frac{2\pi}{4}) = 0 \quad \Omega_2 > \frac{2\pi}{4}$

- سیستم LTI زمان پیوسته ای را با پاسخ فرکانسی $H(j\omega)$ در نظر بگیرید.

$$H(j\omega)=\begin{cases}1, & |\omega|\geq 250\\ 0, & o.w\end{cases}$$

وقتی که ورودی این سیستم $x(t)$ با دوره تناوب اصلی $T=\frac{\pi}{7}$ و ضرایب سری فوریه a_k است. با فرض آنکه ضرایب سری فوریه خروجی $y(t)$ برابر b_k باشد. به ازاء چه مقادیر از K ضرایب سری فوریه خروجی b_k قطعاً صفر است.

$$x(t)=\sum_{k=-\infty}^{\infty} a_k\, e^{jk\left(\frac{2\pi}{\pi/7}\right)t}$$

$$y(t)=\sum_{k=-\infty}^{\infty} \underbrace{a_k\cdot H\left(jk\frac{2\pi}{\pi/7}\right)}\cdot e^{jk\left(\frac{2\pi}{\pi/7}\right)t}$$

ضرایب سری فوریه خروجی $b_k=a_k\cdot H(jk14)$ $\qquad H(j\omega)=\begin{cases}1, & |\omega|\geq 250\\ 0, & o.w\end{cases}$

$$H(jk14)=\begin{cases}1, & |k14|\geq 250\\ 0, & o.w\end{cases} \quad , \quad |k|\geq\frac{250}{14}=17.8$$

$$\boxed{b_k=0, \quad |k|\leq 17}$$

- اطلاعات زیر درباره دنباله $x[n]$ داده شده است.

دنباله $x[n]$ را بصورت sin یا cos تحت شرایط گفته شده بدست آورید.

(a) $x[n]$ is real and odd.

(b) $x[n]$ is periodic with period $N = 6$.

(c) $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = 10$.

(d) $\sum_{n=<N>} (-1)^{n/3} x[n] = 6j$.

(e) $x[1] > 0$.

(a) $\Rightarrow$ a_k مطلقا موهومی و فرد ; $a_k = jc_k$, $c_k = -c_{-k}$

(b) $N=6 \Rightarrow a_k = \frac{1}{6}\sum_{n=<6>} x[n]\, e^{-jk\left(\frac{2\pi}{6}\right)n}$

(c) $\frac{1}{6}\sum_{n=<6>} |x[n]|^2 = \sum_{k=<6>} |a_k|^2 = 10$

(d) $\sum_{n=<6>} \left(e^{-j\pi}\right)^{n/3} x[n] = \sum_{n=<6>} e^{-j\frac{\pi}{3}n}\, x[n] = j6 = 6a_1 \Rightarrow a_1 = j1$

Problem 3 Consider a causal discrete-time LTI system whose input $x[n]$ and output $y[n]$ are related by the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] = x[n] + 2x[n-4]$$

Find the Fourier series representation of the output $y[n]$ when the input is

$$x[n] = 2 + \sin(\pi n/4) - 2\cos(\pi n/2).$$

2

$$H(j\Omega) = \frac{1+2e^{-j4\Omega}}{1-\frac{1}{4}e^{-j\Omega}}$$

$$x[n] = 2 + \frac{1}{2j}e^{j\frac{\pi}{4}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}n} - e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}$$

$$y[n] = 2 \times \frac{3}{3/4} + \frac{1}{2j}\cdot\frac{1+2e^{-j\pi}}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}}\cdot e^{j\frac{\pi}{4}n} - \frac{1}{2j}\cdot\frac{1+2e^{j\pi}}{1-\frac{1}{4}e^{j\frac{\pi}{4}}}\cdot e^{-j\frac{\pi}{4}n}$$

$$- \frac{1+2e^{-j2\pi}}{1-\frac{1}{4}e^{-j\frac{\pi}{2}}}\cdot e^{-j\frac{\pi}{2}n} - \frac{1+2e^{j2\pi}}{1-\frac{1}{4}e^{j\frac{\pi}{2}}}\cdot e^{j\frac{\pi}{2}n}$$

$$= 8 + \frac{1}{2j}\cdot\frac{-1}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}}\cdot e^{j\frac{\pi}{4}n} - \frac{1}{2j}\cdot\frac{-1}{1-\frac{1}{4}e^{j\frac{\pi}{4}}}\cdot e^{-j\frac{\pi}{4}n}$$