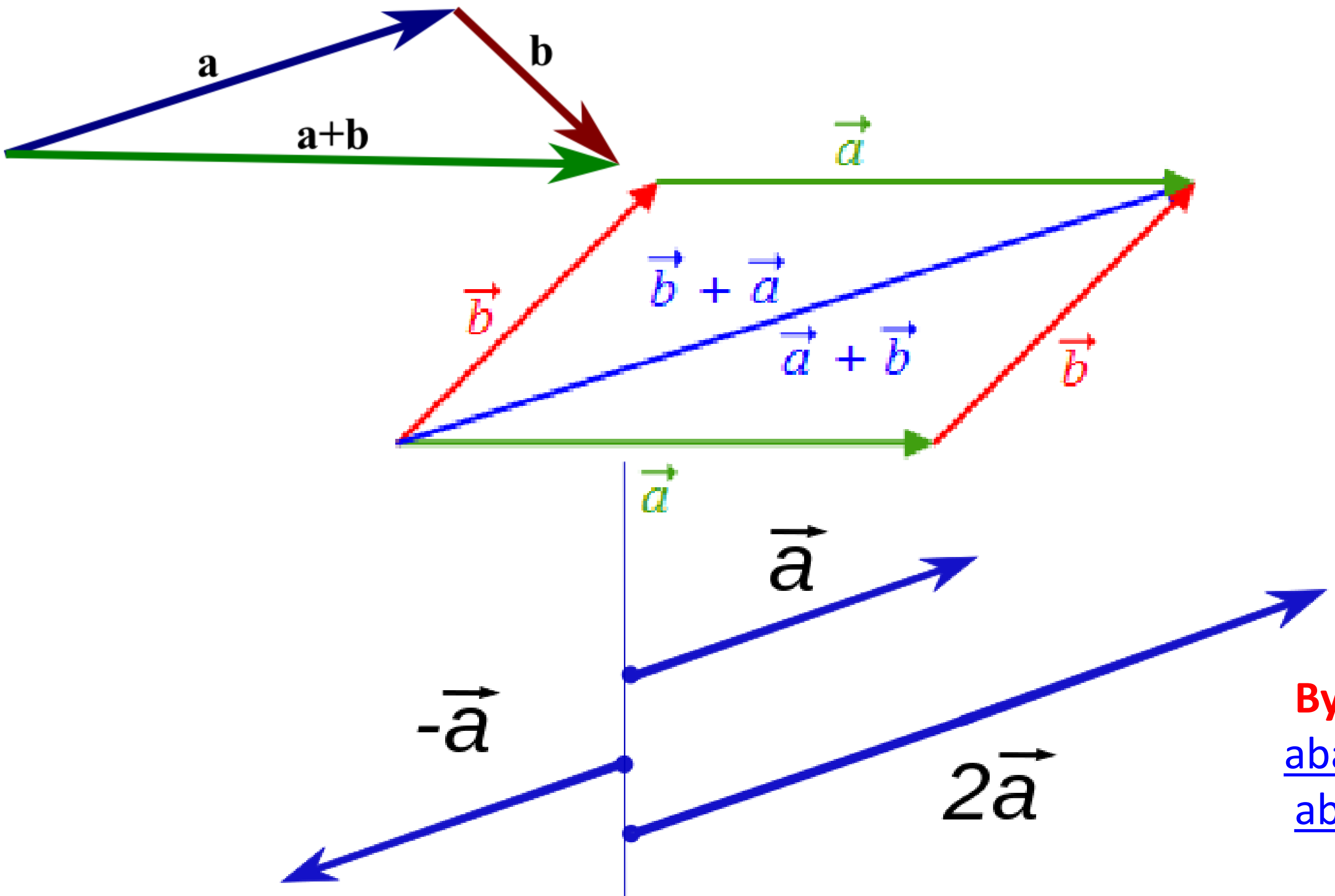
APPLIED MATHEMATICS I

Chapter 1

VECTOR



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Scalar and Vector

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Definition: Scalar is a physical quantity that has only magnitude.

Ex: speed, distance, time, power, work, volume and etc.

Definition: Vector is a physical quantity that has both magnitude and direction.

Ex: displacement, velocity, acceleration and etc.

Vector in R^2 and R^3

Definition: A vector $\vec{u}$ in R^2 (2-space) is any ordered pair of real numbers,

$\vec{u} = (u_1, u_2)$, where $u_1, u_2 \in R$

A vector $\vec{u}$ in R^3 (3-space) is any ordered triple of real numbers,

$\vec{u} = (u_1, u_2, u_3)$, where $u_1, u_2, u_3 \in R$

Vector in R^n $(n - space)$

Definition: An ordered n −tuples of numbers $\vec{u} = (u_1, u_2, u_3, \dots, u_n)$ is called a vector where $u_1, u_2, u_3, \dots, u_n \in R$ (called component of $\vec{u}$) denoted by R^n

Definition: The vector with initial point at the origin is called position vector.

Definition: Any vector with initial point P and terminal point Q, denoted by $\overrightarrow{PQ}$ is called directed vector from P to Q and obtained by: $\overrightarrow{PQ} = Q - P$

Example 1: Find a vector $\vec{u}$ directed from P to Q where $P = (4, -3, 1)$ and $Q = (6,2,5)$

Solution: $\vec{u} = \overrightarrow{PQ} = Q - P$=$(6,2,5) - (4, -3, 1)$= $(2,5,4)$

Example 2: Let $\overrightarrow{PQ} = (6,2,4)$. If the midpoint of the segment $\overline{PQ} = (1,2,3)$, then find the coordinate of P and Q.

Solution: Let $P(x, y, z)$ and $Q(a, b, c)$, then

$$\overrightarrow{PQ} = Q - P = (a - x, b - y, c - z) = (6,2,4)$$
$$\Rightarrow a = x + 6, b = y + 2, c = z + 4 \ldots \ldots \ldots . . (*)$$

From the midpoint of the segment $\overline{PQ}$, we have

$$\overline{PQ} = \frac{P+Q}{2} = \left(\frac{x+a}{2}, \frac{y+b}{2}, \frac{z+c}{2}\right) = (1,2,3)$$

$$\Rightarrow x + a = 2, y + b = 4, z + c = 6 \ldots \ldots \ldots . . (**).$$ using (*) in (**)

$$x + x + 6 = 2, y + y + 2 = 4, z + z + 4 = 6$$

$$\Rightarrow x = -2, y = 1, z = 1 \ldots \ldots \ldots . . (***)$$ substitute (***) in (*),we get

a$= 4, b = 3, c = 5$. As a result, $P = (-2,1,1)$ and Q$= (4,3,5)$

Equality of Vectors

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Definition: Two vectors $\vec{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, v_3, \ldots, v_n)$ iff $u_i = v_i, \ \forall i = 1,2,3, \ldots . n$

Example 3: Find the values of a, b, c if $\vec{u} = (6, a-b, c+3)$ and $\vec{v} = (a+b, -4, 9)$ are equal.

Solution: Since $\vec{u} = \vec{v}$ $\Rightarrow \begin{cases} a+b=6 \\ a-b=-4 \\ c+3=9 \end{cases}$ $\Rightarrow a=1, b=5, c=6$

Addition and Scalar Multiplication

Definition: Let $\vec{u} = (u_1, u_2, u_3, \dots, u_n)$ and $\vec{v} = (v_1, v_2, v_3, \dots, v_n)$ be vectors in R^n, then addition and scalar multiplication is

i. $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3 + \cdots + u_n + v_n$

ii. $k\vec{u} = (ku_1, ku_2, ku_3, \dots, ku_n)$ where $k \in R$

Definition: $\vec{u} // \vec{v} \Leftrightarrow \vec{u} = k\vec{v}$ where $k \in R$

✓ $k > 0$, then $\vec{v}$ and $k\vec{v}$ have the same direction.

✓ $k < 0$, then $\vec{v}$ and $k\vec{v}$ have opposite direction.

Definition: Two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are said to be parallel if there exist scalar k such that $\overrightarrow{PQ} = k\overrightarrow{RS} \Rightarrow Q - P = k(S - R)$

Example 4: If $\vec{u} = (0,2,4)$ and $\vec{v} = (1,3,5)$, then find

a. $\vec{u} + \vec{v}$ b. $2\vec{v}$ c. $\vec{u} - \frac{2}{3}\vec{v}$

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Solution: a. $\vec{u}+\vec{v}=(0+1,2+3,4+5)=(1,5,9)$

b. $\overrightarrow{2u}=(2.0,2.2,2.4)=(0,4,8)$

c. $\vec{u}-\frac{2}{3}\vec{v}=\vec{u}=(0,2,4)-\frac{2}{3}(1,3,5)=(-\frac{2}{3},0,\frac{2}{3})$

Example 5: Determine whether the following set of vectors are parallel or not

a. $\vec{u}=(-8,6,1)$ and $\vec{v}=(-2,3,1)$ b. $\vec{u}=(0,2,4)$ and $\vec{v}=(1,-3,5)$,

Solution: *a.* $\vec{u}//\vec{v}$ since $\vec{u}=4\vec{v}$

b. Not // since $\vec{u}\neq k\vec{v}$

Example 6: Show that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are parallel where $P=(-5,0)$, $Q=(0,5)$, R$=(0,0)$, S$=(5,\ 5)$.

Solution: $\overrightarrow{PQ}=Q-P=(0,5)-(-5,0)=(5,5)$

$\overrightarrow{RS}=S-R=(5,\ 5)-(0,0)=(5,5)$

$\Rightarrow\ \overrightarrow{PQ}\ //\ \overrightarrow{RS}$

Example 7: Let $\vec{u}=(x,4,6)$ and $\vec{v}=(2,y,-3)$, then find x and y such that $\vec{u}\ //\ \vec{v}$

Solution

Since $\vec{u}\ //\ \vec{v}\ \Leftrightarrow \exists k\in R$ such that $\vec{u}=\mathrm{k}\vec{v}$

$\Leftrightarrow (x,4,6)=k(2,y,-3)$

$\Leftrightarrow x=2k, 4=ky,\ 6=-3k$

$\Rightarrow\ k=-2, x=-4, y=-2.$

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Scalar (Dot) Product

Definition: Let $\vec{u} = (u_1, u_2, u_3, \dots, u_n)$ and $\vec{v} = (v_1, v_2, v_3, \dots, v_n)$ be vectors in $\boldsymbol{R}^n$. Then, the scalar or dot product of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}.\vec{v}$ and defined by:

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i$$

Note: The dot product of two vector is scalar.

Example 1: $\vec{u} = (3, -2, 1)$, $\vec{v} = (1, -5, 6)$ and $\vec{w} = (1, 2, 4, 3)$. Then find

a. $\vec{u}.\vec{v}$ b. $\vec{v}.\vec{u}$ c. $\vec{u}.\vec{w}$ d. $(\vec{u}.\vec{v})\,\vec{w}$ e. $(\vec{u} - \vec{v})\,\vec{w}$

Solution: a. $\vec{u}.\vec{v} = 3(1) + (-2)(-5) + 1(6) = 3 + 10 + 6 = 19$

b. $\vec{v}.\vec{u} = 1(3) + (-5)(-2) + 6(1) = 3 + 10 + 6 = 19$

c. $\vec{u}.\vec{w}$ not defined.

d. $(\vec{u}.\vec{v})\,\vec{w} = 19\,(1, 2, 4, 3) = (19, 38, 76, 57)$.

e. $(\vec{u} - \vec{v})\,\vec{w}$ not defined.

Theorem: Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors and $k \in R$.

1. $\vec{u}.\vec{v} = \vec{v}.\vec{u}$
2. $\vec{u}.(\vec{v} + \vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$
3. $(k\vec{u}).\vec{v} = k(\vec{u}.\vec{v}) = \vec{u}.(k\vec{v})$
4. $\vec{u}.\vec{u} \geq 0$ and $\vec{u}.\vec{u} = 0$ iff $\vec{u} = 0$

Definition: Two vectors $\vec{u}$ and $\vec{v}$ are said to be orthogonal or perpendicular iff $\vec{u}.\vec{v} = 0$

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Example 2: Show that the vectors $\vec{u} = (7, -2, 4)$ and $\vec{v} = (2,\ 5, -1)$ are orthogonal.

Solution: $\vec{u}.\vec{v} = 7(4) + (-2)(5) + 4(-1) = 0 \Rightarrow \vec{u} \perp \vec{v}$

Example 3: Find the values of n such that $\vec{u} = (1, -\mathrm{m}, 4, 2)$ and $\vec{v} = (12,\ -5,$

$3, -1)$ are orthogonal.

Solution: since $\vec{u} \perp \vec{v} \Rightarrow \vec{u}.\vec{v} = 0$

$$\Rightarrow (1, -\mathrm{m}, 4, 5).(-12, -5, 3, -1) = 0$$

$$\Rightarrow -12 + 5m + 12 - 5 = 0$$

$$\Rightarrow 5m = 5 \quad \Rightarrow m = 1$$

Example 4: Which of the following pairs of vector are perpendicular?

a. $(3,\ 5,1)$ and $(-2, 1,1)$ b. $(4,1,7)$ and $(2,5, -1)$

Solution: a. $(3,\ 5,1).(-2, 1,1) = 3(-2) + 5(1) + 1(1) = 0 \Rightarrow \vec{u} \perp \vec{v}$

b. $(4,1,7).(2,\ 5, -1) = 4(2) + 1(5) + 7(-1) = 6 \neq 0$

$\Rightarrow$ are not perpendicular.

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Magnitude of a Vector

Definition: The magnitude or length or norm of a vector $\vec{u} = (u_1, u_2, \dots, u_n)$ denoted by $\|\vec{u}\|$ and given by: $\|\vec{u}\| = \sqrt{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}$

Example 1: Find the norm of a vector $\vec{u} = (3, -5, 2)$

Solution: $\|\vec{u}\| = \sqrt{3^2 + (-5)^2 + 2^2} = \sqrt{38}$.

Example 2: Suppose $\|\vec{u}\| = 8$, find a such that $\vec{u} = (-3, a, 2, -4)$

Solution: $\|\vec{u}\| = \sqrt{(-3)^2 + (a)^2 + 2^2 + (-4)^2} = 8$

$$\Rightarrow (-3)^2 + (a)^2 + 2^2 + (-4)^2 = 64$$

$$\Rightarrow 9 + a^2 + 4 + 16 = 64$$

$$\Rightarrow a^2 + 29 = 64$$

$$\Rightarrow a^2 = 35 \qquad \Rightarrow a = \pm\sqrt{35}$$

Theorem: Let $\vec{u}$ and $\vec{v}$ be two vectors in R^n and $k \in R$.

i. $\|\vec{u}\| \geq 0$ ii. $\|\vec{u}\| = \|-\vec{u}\|$ iii. $\|k\vec{u}\| = k\|\vec{u}\|$

iv. $\|\vec{u} \pm \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \pm 2\|\vec{u}\|\,\|\vec{v}\|$

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Definition: The distance between two points $P(x_1, x_2, \ldots, x_n)$ and $Q(y_1, y_2, \ldots, y_n)$ Is given by:

$$\|\overrightarrow{PQ}\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

Example 3: Let $P(-2, -7, 4)$ and $Q(-1, 3, -5)$, then find the distance between P and Q.

Solution: $\|\overrightarrow{PQ}\| = \sqrt{(-1-(-2))^2 + (3-(-7))^2 + (-5-4)^2}$

$$= \sqrt{1 + 100 + 81}$$

$$= \sqrt{182}$$

Definition: A vector whose magnitude is 1 is called unit vector.

Or vector $\vec{u}$ is a unit vector iff $\|\vec{u}\| = 1$.

Example 4: Vectors:- $(0, -1, 0, 0)$, $(1, 0)$, $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ are some ex of unit vectors

Note: The process of multiplying a vector $\vec{u}$ by the reciporical of its magnitude to get a unit vector with same direction is called normalizing of $\vec{u}$.

Definition: $\frac{1}{\|\vec{u}\|}\vec{u}$ is a unit vector in the direction of $\vec{u}$

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Definition: For vector $\vec{u}$

i. $\vec{v} = \frac{\vec{u}}{\|\vec{u}\|}$ is a unit vector in the same direction as $\vec{u}$.

ii. $\vec{v} = -\frac{\vec{u}}{\|\vec{u}\|}$ is a unit vector in the opposite direction as $\vec{u}$.

Remark: $\vec{u} = (u_1, u_2, u_3) = (u_1, 0, 0) + (0, u_2, 0) + (0, 0, u_3)$

$= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$

$= u_1 i + u_2 j + u_3 k$

Here, the vectors $\mathrm{i} = (1,0,0)$, $\mathrm{j} = (0,1,0)$ and $\mathrm{k} = (0,0,1)$ is called standard or basis unit vector.

Example 5: $\vec{u} = -3i + j - k$ and $\vec{v} = -2j + 4k$. Then find a unit vector in the same direction of $\vec{u} + \vec{v}$

Solution: $\frac{\vec{u}+\vec{v}}{\|\vec{u}+\vec{v}\|} = \frac{-3i-j+3k}{\sqrt{(-3)^2+(-1)^2+3^2}} = \frac{1}{\sqrt{19}}(-3i - j + 3k)$

Example 6: If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, then find $\left\|\frac{1}{3}\vec{u} - 2\vec{v}\right\|$

Solution: Since $\vec{u}$ and $\vec{v}$ are orthogonal unit vector $\Rightarrow \|\vec{u}\| = \|\vec{u}\| = 1, \vec{u}.\vec{v} = 0$. Then,

$$\left\|\frac{1}{3}\vec{u} - 2\vec{v}\right\| = \sqrt{\left(\frac{1}{3}\vec{u} - 2\vec{v}\right).\left(\frac{1}{3}\vec{u} - 2\vec{v}\right)} = \sqrt{\frac{1}{9}\vec{u}.\vec{u} + 4\vec{v}.\vec{v}} = \sqrt{\frac{37}{9}}$$

Example 7: Let $\vec{u} = -5i + \sqrt{3}j - 2k$. Then find

a. The unit vector in the same direction of $\vec{u}$

b. The unit vector in the opposite direction of $\vec{u}$

Solution: Here, $\|\vec{u}\| = \sqrt{(-5)^2 + (\sqrt{3})^2 + (-2)^2}$

$$= \sqrt{25 + 3 + 4}$$

$$= 4\sqrt{2}$$

a. $\vec{v} = \frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{4\sqrt{2}}(-5i + \sqrt{3}j - 2k)$

b. $\vec{v} = -\frac{1}{\|\vec{u}\|}\vec{u} = -\frac{1}{4\sqrt{2}}(-5i + \sqrt{3}j - 2k)$

Remark: All unit vector in R^2 are of the form $(cos\theta, sin\theta)$, where $\theta \in [0,\ 2\pi]$.

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Angle Between Two Vectors

Theorem: If $\vec{u}$ and $\vec{v}$ are non-zero vectors in R^2 or R^3 and if θ is the angle between them, then

$$\vec{u}.\vec{v} = \|\vec{u}\|\|\vec{v}\|cos\theta$$

That is

$$cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\Rightarrow \theta = cos^{-1}\left(\frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right), \theta \in [0, \pi]$$

Definition: Two non zero vectors $\vec{u}$ and $\vec{v}$ are said to be orthogonal or perpendicular iff

$\vec{u}.\vec{v} = 0$ i.e $\theta = \frac{\pi}{2}$

Example 1: Let $\|\vec{u}\| = 3$, $\|\vec{v}\| = \sqrt{2}$ and $\vec{u}.\vec{v} = 5$. Find the angle between $\vec{u}$ and $\vec{v}$

Solution :

$$cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\| \ \|\vec{v}\|} = \frac{5}{3\sqrt{2}} \qquad \Rightarrow \theta = cos^{-1}\left(\frac{5}{3\sqrt{2}}\right)$$

Example 2: Find the angle between vectors $\vec{u} = i - 2j$ and $\vec{v} = i - k$

$$\|\vec{u}\| = \sqrt{5} , \|\vec{v}\| = \sqrt{2}, \vec{u}.\vec{v} = 1$$

$$cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\| \ \|\vec{v}\|} = \frac{1}{\sqrt{5}\sqrt{2}} \Rightarrow \theta = cos^{-1}\left(\frac{1}{\sqrt{5}\sqrt{2}}\right)$$

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Example 3: Find the angle between the vector $\sqrt{2}i - j + 3k$ and

a. positive x- axis b. negative y- axis

Solution: a. Let $\vec{u} = \sqrt{2}i - j + 3k$ and $\vec{v} = mi$ where $m > 0$ be +ve x-axis

Here, $\vec{u}.\vec{v} = m\sqrt{2}$, $\|\vec{u}\| = 2\sqrt{3}$, $\|\vec{v}\| = m$

$$\theta = cos^{-1}\left(\frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = cos^{-1}\left(\frac{m\sqrt{2}}{2\sqrt{3}m}\right) = cos^{-1}\left(\frac{\sqrt{2}}{2\sqrt{3}}\right)$$

b. Let $\vec{u} = \sqrt{2}i - j + 3k$ and $\vec{v} = -mj$ where $m > 0$ be -ve y-axis

Here, $\vec{u}.\vec{v} = -m$, $\|\vec{u}\| = 2\sqrt{3}$, $\|\vec{v}\| = m$

$$\theta = cos^{-1}\left(\frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = cos^{-1}\left(\frac{-m}{2\sqrt{3}m}\right) = cos^{-1}\left(\frac{-1}{2\sqrt{3}}\right)$$

Theorem: Let $\vec{u}$ and $\vec{v}$ are non zero vectors, then

a. $\|\vec{u} \pm \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \pm \|\vec{u}\|\|\vec{v}\|cos\theta$

b. $\|\vec{u} \pm \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \Leftrightarrow \vec{u}.\vec{v} = 0$

c. $\|\vec{u} + \vec{v}\| = \|\vec{u} - \vec{v}\| \Leftrightarrow \vec{u}.\vec{v} = 0$

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Example 4:

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors with $\vec{u}+\vec{v}+\vec{w}=0$, $\|\vec{u}\|=1$, $\|\vec{v}\|=2$, $\|\vec{u}\|=3$

Find $\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{v}.\vec{w}$

Solution: $\vec{u}+\vec{v}+\vec{w}=0$

$$\Rightarrow \|\vec{u}+\vec{v}+\vec{w}\| = \|0\|$$

$$\Rightarrow (\vec{u}+\vec{v}+\vec{w}).(\vec{u}+\vec{v}+\vec{w}) = 0$$

$$\Rightarrow \vec{u}.\vec{u}+\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{u}.\vec{v}+\vec{v}.\vec{v}+\vec{v}.\vec{w}+\vec{u}.\vec{w}+\vec{v}.\vec{w}+\vec{w}.\vec{w}=0$$

$$\Rightarrow \|\vec{u}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2+2(\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{v}.\vec{w})=0$$

$$\Rightarrow 1+4+9+2(\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{v}.\vec{w})=0$$

$$\Rightarrow 2(\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{v}.\vec{w})=-14$$

$$\Rightarrow \vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{v}.\vec{w}=-7$$

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Cross Product

Definition: Let $\vec{u} = u_1 i + u_2 j + u_3 k$ and $\vec{v} = v_1 i + v_2 j + v_3 k$ be vectors in R^3. The cross or vector product of $\vec{u}$ and $\vec{v}$ is the vector given by:

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)i - (u_1 v_3 - u_3 v_1)j + (u_1 v_2 - u_2 v_1)k$$

Note:

1. The cross product of two vectors $\vec{u}$ and $\vec{v}$ is the vector .
2. $\vec{u} \times \vec{v}$ is defined only when $\vec{u}$ and $\vec{v}$ are in R^3.
3. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
4. $\vec{u} \times \vec{v} = 0 \quad iff \quad \vec{u} // \vec{v}$.

Example 1: If $\vec{u} = i - 5j + 3k$ and $\vec{v} = 4i - k$, then find

a. $\vec{u} \times \vec{v}$ b. $\vec{v} \times \vec{u}$ c. $\vec{u} \times \vec{u}$

Solution:

a. $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & -5 & 3 \\ 4 & 0 & -1 \end{vmatrix} = (5 - 0)i - (-1 - 12)j + (0 + 20)k = 5i + 13j + 20k$

b. $\vec{v} \times \vec{u} = \begin{vmatrix} i & j & k \\ 4 & 0 & -1 \\ 1 & -5 & 3 \end{vmatrix} = (0 - 5)i - (12 + 1)j + (-20 - 0)k$

$= -5i - 13j - 20k$

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c. $\vec{u} \times \vec{u} = \begin{vmatrix} i & j & k \\ 1 & -5 & 3 \\ 1 & -5 & 3 \end{vmatrix} = (-15+15)i - (3-3)j + (-5+5)k = 0i + 0j + 0k$

Theorem Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be any vectors in R^3 and $k \in R$, then

i. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

ii. $\vec{u} \times \vec{u} = 0$

iii. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

iv. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$

$v.$ $(\vec{u} \times \vec{v})^2 = (\vec{u}.\vec{u})(\vec{v}.\vec{v}) - (\vec{u}.\vec{v})^2$

$vi.$ $\vec{u}.(\vec{v} \times \vec{w}) = \vec{w}.(\vec{u} \times \vec{v})$

Example 2: Let $\vec{u} = i - 3j - 2k$ and $\vec{v} = i - k$. Then find vector $\vec{w}$ with norm 4 in the opposite direction to the vector $\vec{u} \times 3\vec{v}$

Here, $\vec{u} \times 3\vec{v} = \begin{vmatrix} i & j & k \\ 1 & -3 & -2 \\ 3 & 0 & -3 \end{vmatrix} = 9i - 3j + 9k$

Then, the vector the norm 4 in the opposite direction to $\vec{u} \times 3\vec{v}$ is

$\vec{w} = -4\vec{b}$ where $\vec{b} = \frac{\vec{u} \times 3\vec{v}}{\|\vec{u} \times 3\vec{v}\|} = \frac{1}{\sqrt{171}}(9i - 3j + 9k)$

Thus, $\vec{w} = \frac{-4}{\sqrt{171}}(9i - 3j + 9k)$

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Area

Definition: Let $\vec{u}$ and $\vec{v}$ be non zero and non parallel vectors in R^3.

1. The area of parallelogram with sides $\vec{u}$ and $\vec{v}$ is given by:

$$A = \|\vec{u} \times \vec{v}\|$$

2. The area of triangle with sides $\vec{u}$ and $\vec{v}$ is given by:

$$A = \frac{1}{2}\|\vec{u} \times \vec{v}\|$$

Volume

Definition: Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ do not lie in the same plane, then the volume of parallelepiped with edges $\vec{u}$, $\vec{v}$ and $\vec{w}$ is given by.

$$\mathrm{V} = |\vec{u}.(\vec{v} \times \vec{w})| = |\vec{v}.(\vec{w} \times \vec{u})| = |\vec{w}.(\vec{u} \times \vec{v})|$$

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Example: Let $\vec{u} = -i - j + k$, $\vec{v} = 3j + 2k$ and $\vec{w} = 2i - j + 5k$. Then,

a. Find the area of parallelogram determined by $\vec{u}$ and $\vec{v}$.

b. Find the area of triangle determined by $\vec{u}$ and $\vec{v}$.

c. Find the volume of parallelepiped whose adjacent sides are $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Solution: $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ -1 & -1 & 1 \\ 0 & 3 & 2 \end{vmatrix} = -5i + 2j - 3k$ and $\|\vec{u} \times \vec{v}\| = \sqrt{38}$

a. The area of parallelogram is $\sqrt{38}$ square unit.

b. The area of triangle is $\frac{1}{2}\sqrt{38}$ square unit.

c. $\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 0 & 3 & 2 \\ 2 & -1 & 5 \end{vmatrix} = 17i + 4j - 6k$

Then, volume of parallelepiped is

$$V = |\vec{u}.(\vec{v} \times \vec{w})| = |(-i - j + k).(17i + 4j - 6k\ )|$$

$$= |-17 - 4 + 6| = 15 \text{ cubic unit}$$

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Equation of a Line

Definition: Let $P_0 = (x_0, y_0, z_0)$ be a given point on a line l and $P = (x, y, z)$ be any arbitrary point on l. If $\vec{v} = (a, b, c)$ is parallel to l, then

1. The vector equation of l is written as

$$P = P_0 + t\vec{v}, \text{ where } t \in R$$

2. The parametric equation of a line l is given by

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct, \text{ where } t \in R$$

3. The symmetric equation of a line l is given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example: Find the parametric and symmetric equation of a line passing through the point $(1, -2, 3)$ and parallel to the vector $(-6, 4, 1)$.

Solution: Let $P(x, y, z)$ be any arbitrary point on a line l passing through a point $P_0 = (1, -2, 3)$.

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Thus, $\overrightarrow{P_0P} = t\vec{v}$ where $\vec{v}// \, l$ and t is parameter.

$$\Rightarrow (x-1, y+2, z-3) = t(-6,4,1)$$

$$\Rightarrow \quad \left.\begin{array}{c} x = 1 - 6t \\ y = -2 + 4t \\ z = 3 + t \end{array}\right\}$$ Is the parametric equation of a line l.

The symmetric equation of a line l is

$$\frac{x-1}{-6} = \frac{y+2}{4} = \frac{z-3}{1}$$

Example: Find the parametric equation of a line passes through the point $(2,-1,7)$ and $(3,1,-4)$.

Solution: Let $\mathrm{P} = (2,-1,7)$ and $\mathrm{Q} = (3,1,-4)$. First we need to find a direction vector $\vec{v} = \overrightarrow{PQ} = (1,2,-11)$.

If we take $P_0 = (2, \ -1,7)$ the parametric equation will be $x = 2 + t, y = -1 + 2t, z = 7 - 11t$

If we take $P_0 = (3, \ 1,-4)$ the parametric equation will be $x = 3 + t, y = 1 + 2t, z = -4 - 11t$

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Example: Find the equation of a line passes through the point $(5, -1, 3)$ and parallel to $\vec{v} = (1, 2, 0)$.

Solution: Let $\mathrm{P} = (x, y, z)$ be any arbitrary point on a line passing through $P_0 = (5, -1, 3)$

Thus, $P = P_0 + \mathrm{t}\vec{v}$

$$(x, y, z) = (5, -1, 3) + t(1, 2, 0).$$

$$\Rightarrow \quad \left.\begin{aligned} x &= 5 + t \\ y &= -1 + 2t \\ z &= 3 \end{aligned}\right\}$$ Is the parametric equation of a line l.

The symmetric equation of a line l is

$$\frac{x-5}{1} = \frac{y+1}{2} = \frac{z-3}{0}$$

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1. Given $\vec{a} = i$ and $\vec{b} = i + j$. Then find the value of k such that $\vec{a} + k\vec{b}$ is orthogonal to $\vec{a}$.

Solution: $(\vec{a} + k\vec{b}).\vec{a} = 0$

$$\vec{a}.\vec{a} + k\vec{b}.\vec{a} = 0$$

$$i.i + k(i + j).i = 0$$

$$1 + k(i.i + j.i) = 0$$

$$1 + k(1 + 0) = 0$$

$$1 + k = 0$$

$$k = -1$$

2. Find the angle between the vectors $2i - 3j + 4k$ and $3i - 4j + k$.

Solution: Let $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = 3i - 4j + k$.

$$\|\vec{a}\| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}, \quad \|\vec{b}\| = \sqrt{3^2 + (-4)^2 + 1^2} = \sqrt{26}$$

$$\vec{a}.\vec{b} = 22$$

$$\theta = cos^{-1}\left(\frac{\vec{a}.\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) = cos^{-1}\left(\frac{22}{\sqrt{29}\sqrt{26}}\right)$$

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3. Find the angle θ between the vectors $\sqrt{6}i + j - k$ and positive x −axis.

Solution: Let $\vec{a} = \sqrt{6}i + j - k$ and $\vec{b} = b_1 i$, where b_1 be positive x −axis.

$$\|\vec{a}\| = \sqrt{(\sqrt{6})^2 + 1^2 + (-1)^2} = 2\sqrt{2}, \quad \|\vec{b}\| = \sqrt{(b_1)^2 + 0^2 + 0^2} = b_1$$

$$\vec{a}.\vec{b} = \sqrt{6}\, b_1 \qquad \theta = cos^{-1}\left(\frac{\sqrt{6}\, b_1}{2\sqrt{2}\, b_1}\right) = cos^{-1}\left(\frac{\sqrt{6}}{2\sqrt{2}}\right)$$

4. Let $\vec{a} = i + j - k$ and $\vec{b} = 2i - j + 2k$ and $\vec{c} = 2i - j + k$. Then find

a. $\|2\vec{a} - \vec{b}\|$ b. $(\vec{a} - 2\vec{c}) \times \vec{b}$ c. $(\vec{a} + 2\vec{b}).\vec{c}$

Solution: a. $2\vec{a} - \vec{b} = 2(i + j - k) - (2i - j + 2k) = 3j - 4k$

$$\|2\vec{a} - \vec{b}\| = \|3j - 4k\| = \sqrt{3^2 + (-4)^2} = 5$$

b. $\vec{a} - 2\vec{c} = i + j - k - 2(2i - j + k) = -3\mathrm{i} + 3\mathrm{j} - 3\mathrm{k}$

$$\text{b. } (\vec{a} - 2\vec{c}) \times \vec{b} = \begin{vmatrix} i & j & k \\ -3 & 3 & -3 \\ 2 & -1 & 2 \end{vmatrix} = 3i - 3k$$

c. $\vec{a} + 2\vec{b} = i + j - k + 2(2i - j + 2k) = 5i - j + 3k$

$$(\vec{a} + 2\vec{b}).\vec{c} = (5i - j + 3k).(2i - j + k) = 14$$

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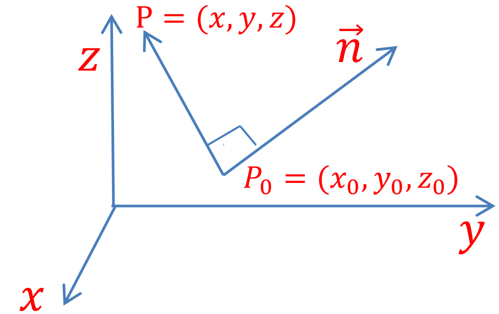
Equation of a Plane

Definition: The plane containing the point $P_0 = (x_0, y_0, z_0)$ and having normal vector $\vec{n} = (a, b, c)$ can be represented by the standard form of the equation of a plane z



$$a(x - x_0) + b(y - y_0) + \mathrm{c}(z - z_0) = 0$$

Remark: 1. The general form of the equation of a plane in space is given by:

$$ax + by + \mathrm{cz} + \mathrm{d} = 0, \text{ where } \mathrm{d} = -(\mathrm{a}x_0 + \mathrm{b}y_0 + cz_0)$$

2. A vector orthogonal to the plane is called normal vector.

Example: Find equation of a plane that contains point $(-2,0,6)$ and is normal to $(1,2,5)$.

Solution: Let $\mathrm{P} = (x, y, z)$ be any arbitrary point in the plane which contains a point $P_0 = (-2,0,6)$ and having normal vector $\vec{n} = (1,2,5)$

$$\vec{n}.\overrightarrow{P_0P} = 0$$

$$\Rightarrow \quad (1,2,5).(x + 2, y, z - 6) = 0$$

$$x + 2y + 5z - 28 = 0$$

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Example: Find equation of a plane that contains point $(2,3,-5)$ and is orthogonal to y-axis.

Solution: Let $\mathrm{P} = (x, y, z)$ be any arbitrary point in the plane which contains a point $P_0 = (2,3,-5)$ and having normal vector $\vec{n} = (0,1,0)$ since y-axis is perpendicular to the plane.

$$\vec{n}.\overrightarrow{P_0P} = 0$$

$$\Rightarrow \quad (0,1,0).(x-2, y-3, z+5) = 0$$

$$y - 3 = 0$$

$$\Rightarrow \quad y = 3$$

Example: Find the equation of the plane that contains points $(-1,0,1)$, $(2,1,1)$ and $(-2,0,3)$.

Solution: Let $\mathrm{P} = (-1,0,1)$, $\mathrm{Q} = (2,1,1)$ and $\mathrm{R} = (-2,0,3)$.

Then, we need to find a vector normal to the plane. Now,

$$\overrightarrow{PQ} = Q - P = (3,1,0)$$

$$\overrightarrow{QR} = R - Q = (-4,-1,2)$$

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$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ -4 & -1 & 2 \end{vmatrix} = 2i - 6j + k$$

Then, let $P = (x, y, z)$ be any arbitrary point in the plane which contains a point $P_0 = (-1,0,1)$ and normal vector $\vec{n} = (2, -6, 1)$.

$$\vec{n}.\overrightarrow{P_0P} = 0$$

$$\Rightarrow \quad (2, -6, 1).(x + 1, y, z - 1) = 0$$

$$2(x + 1) - 6y + (z - 1) = 0$$

$$2x - 6y + z + 1 = 0$$

Example: Find equation of plane and z-intercept of equation of the plane through the point $(4, -1, -2)$ with normal vector $\vec{n} = (1,2,3)$.

An equation of plane: $(x - 4) + 2(y + 1) + 3(z + 2) = 0$

$$\Rightarrow x + 2y + 3z + 4 = 0$$

To find z-intercept of the plane, we set $x = y = 0$

$\Rightarrow 3z + 4 = 0 \Rightarrow z = -\dfrac{4}{3}$ Thus, $(0,0,-\frac{4}{3})$ is z-intercept of the plane

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Parallel and Perpendicular Plane

Definition: 1. Two planes are said to be parallel if their normal are parallel.

2. Two planes are said to be perpendicular if their normal are perpendicular.

3. The angle between two planes are the angle between their normal.

Example: Show that the planes $\pi_1: x - 2y + 4z = 0$ and $\pi_2: 2x - 4y + 8z = 0$ are parallel.

Solution: Let $\vec{n}_1 = (1, -2,4)$ and $\vec{n}_2 = (2, -4,8)$ are normal vectors of π_1 and π_2 respectively.

$$\vec{n}_2 = 2\vec{n}_1 \qquad \Rightarrow \vec{n}_1 // \vec{n}_2 \qquad \therefore \pi_1 // \pi_2$$

Example: Let $\pi_1: 8x - 2y + z = 0$ and $\pi_2: x + 4y = 0$. Show that $\pi_1 \perp \pi_2$

Solution: $\vec{n}_1 = (8, -2,1)$ and $\vec{n}_2 = (1,4,0)$

$$\vec{n}_1 . \vec{n}_2 = (8, -2,1).(1,4,0)$$

$$= 8(1) + (-2)(4) + 1(0)$$

$$= 0$$

$$\Rightarrow \vec{n}_1 \perp \vec{n}_2 \qquad \therefore \pi_1 \perp \pi_2$$

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Example: Let $\pi_1: x - y + 3z = 0$ and $\pi_2: 2x - 3y + z - 7 = 0$.

Find the angle between π_1 and π_2.

Solution: $\vec{n}_1 = (1, -1, 3)$ and $\vec{n}_2 = (2, -3, 1)$

So that $\vec{n}_1$ is neither $\perp$ nor // to $\vec{n}_2$, hence, there is an angle θ between $\vec{n}_1$ and $\vec{n}_2$

$$\vec{n}_1 . \vec{n}_2 = (1, -1, 3) . (2, -3, 1) = 8$$

$$\|\vec{n}_1\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$$

$$\|\vec{n}_2\| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$$

Hence, $\theta = cos^{-1}\left(\frac{\vec{n}_1 . \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|}\right)$

$$= cos^{-1}\left(\frac{8}{\sqrt{11}\sqrt{14}}\right)$$

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Intersection of two lines in space

Definition: Let $l_1: \begin{cases} x = x_1 + a_1 t \\ y = y_1 + b_1 t \\ z = z_1 + c_1 t \end{cases}$ and $l_2: \begin{cases} x = x_2 + a_2 r \\ y = y_2 + b_2 r \\ z = z_2 + c_2 r \end{cases}$

Are parametric equation of a lines l_1 and l_2.

To find intersection point (if any) of l_1 and l_2 equate the corresponding equation

$$\begin{cases} x_1 + a_1 t = x_2 + a_2 r \\ y_1 + b_1 t = y_2 + b_2 r \\ z_1 + c_1 t = z_2 + c_2 r \end{cases}$$

Note: We use the ff steps to find intersection of two lines:

1. Write the equation of each lines in parametric form(if they are not)
2. Equate the parametric equation and solve for t and r.
3. Put the values of t and r in each line to find the points P and Q.
4. If $P = Q$, the lines intersect and if $P \neq Q$, the lines never intersect.

Remark: Two lines in space that are neither parallel nor intersecting are called skew lines

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Example: Show that the lines l_1 and l_2 intersect, and find their point of intersection

$$l_1: x = 2 + t, y = 2 + 3t, z = 3 + t$$

$$l_2: x = 2 + r, y = 3 + 4r, z = 4 + 2r$$

Solution: To find intersection point of l_1 and l_2 equate the corresponding parametric equation. i.e

$$\begin{cases} 2 + t = 2 + r \\ 2 + 3t = 3 + 4r \\ 3 + t = 4 + 2r \end{cases} \quad \begin{array}{l} \Rightarrow t = r \\ \Rightarrow 2 + 3t = 3 + 4t \\ \Rightarrow t = r = -1 \end{array}$$

Hence, if we put $t = -1$ and $r = -1$ in l_1 and l_2 , we obtain $P = (1, -1,2)$ and $\mathrm{Q} = (1, -1,2)$

Hence, $\mathrm{P} = \mathrm{Q} = (1, -1,2)$ and the two lines intersect at $(1, -1,2)$.

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